On regular induced subgraphs of edge-regular graphs

Rhys J. Evans

Queen Mary
University of London

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Declaration

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Details of collaboration and publications: parts of this work have been completed in collaboration with Sergey Goryainov and Dmitry Panasenko. A large part of this work is found in the paper [31]. Throughout this thesis, we attempt to make transparent whether a result was known previous to the current work, found through a collaborative effort, or determined solely by the current author.

Abstract

We study edge-regular graphs and their regular induced subgraphs. More precisely, we are interested in vertex partitions of edge-regular graphs into two parts, for which one or both of the parts induce a regular subgraph. We can divide the thesis into three major components, where we focus on different subclasses of edge-regular graphs and partitions with varying properties.

First we consider regular induced subgraphs of strongly regular graphs. We determine new upper and lower bounds on the order of a d-regular induced subgraph of a strongly regular graph with given parameters. We prove our bounds are at least as good as some well-known bounds for the order of regular induced subgraphs of regular graphs. Further, we find that our bounds are often better than these well-known bounds.

Secondly, we investigate edge-regular graphs which have a partition into a clique and a regular induced subgraph. We first explore some fundamental properties of these graphs and their parameters. Then we construct new graphs having partitions with previously unseen properties, answering questions found in the literature.

Finally, we examine partitions of the Johnson graphs $J(n, 3)$ into two regular induced subgraphs. We use the combinatorial structure and spectrum of $J(n, 3)$ to investigate the local structure of particular partitions in $J(n, 3)$.

To study these problems, we use algebraic and computational tools in conjunction with combinatorial arguments. A manual for AGT, a software package developed and used during this thesis, is found in the appendix.

Acknowledgements

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I would also like to thank Sergey Goryainov for providing me with ample opportunities to travel and collaborate.

To Charlie,

Contents

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Chapter 1

Introduction

1.1 Overview

The question of finding the maximum order of a d-regular induced subgraph of a given graph Γ is a generalisation of many problems in graph theory, for example, finding the independence number and the order of a maximum induced matching in Γ. In general, finding a d-regular induced subgraph of a given graph Γ is computationally hard (see Asahiro et al. [3]).

For any regular graph Γ, a partition of its vertices for which both parts induce a regular graph has other equivalent definitions in the literature, namely equitable 2-partitions and (d, m)-regular sets. Such partitions are usually computationally hard to find, but have nice spectral properties.

In Chapter 2 we introduce a hierarchy of graph classes. Due to the complexity of finding induced regular subgraphs, we focus our attention on specific classes of graphs. Strongly regular graphs and distance-regular graphs have been a fruitful area of study because of their combinatorial and algebraic properties. Edge-regular graphs include all distance-regular graphs, but do not have any obvious algebraic properties. This creates interest in extremal properties which force edge-regular graphs to be distance-regular.

In Section 2.4 we introduce intersection numbers and a certain block intersection polynomial. These tools are derived via double counting arguments, and have been used to analyse designs, graphs and digraphs. Known global properties of a graph can often be applied with a block intersection polynomial to reveal restrictions on its structure. In Section 2.5 we focus on m-regular sets, and give a characterisation of m-regular sets in terms of their intersection numbers and their corresponding block intersection polynomial (Corollary 2.5).

In Chapter 3, we generalise certain results of Gary Greaves and Leonard Soicher [38]. We use a certain block intersection polynomial to determine new upper and lower bounds on the order of a d-regular induced subgraph of any strongly regular graph with given parameters. For strongly regular graphs, our new bounds are at least as good as the bounds on the order of a d-regular induced subgraph of a k-regular graph determined by Willem Haemers [40] (Theorem 3.11). For small strongly regular graphs, computations suggest our bounds improve on Haemers' bounds relatively often, and the improvement can be quite large. In Section 3.4.2, a considerable amount of work is done to show that for each non-negative integer d, our new upper bound beats the upper bound of Haemers for infinitely many strongly regular graphs.

Chapters 4 and 5 contain results on edge-regular graphs with regular cliques (a large part of which form the content of the paper of Sergey Goryainov, Dmitry Panasenko and the author [31]). The study of these graphs was originally motivated by a problem of Arnold Neumaier, who asked if there exists an edge-regular, non-strongly regular graph containing a regular clique. Thus, a graph Γ is a Neumaier graph with parameters $(v, k, \lambda; m, s)$ if it is non-complete, edge-regular with parameters (v, k, λ), and it contains an m-regular s-clique. Further, Γ is a strictly Neumaier graph if it is a non-strongly regular Neumaier graph.

In Chapter 4 we collect results on the parameters of Neumaier graphs. We present conditions which must hold for any Neumaier graph, and conditions which force a Neumaier graph to be strongly regular. In particular, we show a Neumaier graph Γ must have parameters $(v, k, \lambda; m, s)$ which satisfy the inequality

$$k - \lambda - s + m - 1 \geq 0$$

(Theorem 4.10), and equality holds if and only if Γ is in one of three well-known families of strongly regular graphs (Theorem 4.11). In Section 4.4 we use the results of the previous sections and computational tools to determine the smallest strictly

Neumaier graph, which has parameters $(16, 9, 4; 2, 4)$. We finish the chapter by giving the adjacency matrices of some small strictly Neumaier graphs.

In Chapter 5, we concentrate on constructing strictly Neumaier graphs. We start by generalising the approaches of Gary Greaves and Jack Koolen [36,37], and give a few examples of strictly Neumaier graphs found using our generalisation. In Section 5.2 we construct two infinite families of strictly Neumaier graphs by applying a switching operation to the graphs of a family of strongly regular graphs. For every integer $i \geq 2$, each of these families contain exactly one strictly Neumaier graph having a 2^{i-1}-regular 2^i-clique.

In Chapter 6, we go about enumerating the equitable 2-partitions of the Johnson graphs $J(n, 3)$. We start by reviewing the known equitable 2-partitions of $J(n, 3)$, where we focus on the constructions most relevant to our work. Intersection numbers are used to prove the nonexistence of an equitable 2-partition for which at least one part induces a subgraph of diameter at most 2 (Theorem 6.8). An algebraic tool is also used to investigate the local structure of particular equitable 2-partitions in Section 6.5. We use this approach to prove certain local structures cannot occur when $n > 8$.

Software for symbolic computation and analysis of graphs has been used throughout the thesis (see Section 1.2). For example, several results used to compare bounds on the order of regular induced subgraphs in Chapter 3 are verified using Maple [8]. Collections of graphs have also been used to test conjectures and find examples of small graphs with specific properties. In particular, the discovery of many small strictly Neumaier graphs in Chapter 4 came from experiments using GAP [39].

In Chapter 7 we summarise the results found in this thesis and consider some possible further research. This includes several questions about the structure of an edge-regular, non-strongly regular graph containing a regular clique, and the complete enumeration of the equitable 2-partitions of the Johnson graphs $J(n, 3)$.

1.2 Software

Throughout the thesis, we reference the software used to investigate the objects of study. Most of the presented computations are carried out in the computer algebra system GAP [39].

To deal with graphs and block intersection polynomials, we rely heavily on the GAP packages GRAPE [55] and DESIGN [56]. These packages form the basis of our new GAP package, AGT.

The AGT package contains methods used for the determination of various algebraic and regularity properties of graphs, as well as certain substructures of graphs. The package also contains a library of small strongly regular graphs, which has turned out to be a useful resource for computational experiments. AGT is now part of the official distribution of GAP version 4.11.0, and can be found at https://gap-packages.github.io/agt/.

When we use GAP for investigations within the text, we will use the following environment which reflects the computations within a session of GAP in a terminal.

```
gap> gamma:=SRG([16,9,4,6],1);
rec( adjacencies := [ [ 8, 9, 10, 11, 12, 13, 14, 15, 16 ] ],
group := Group([ (6,7)(9,10)(12,13)(15,16),
(5,6)(8,9)(11,12)(14,15), (2,5)(3,6)(4,7)(9,11)(10,14)(13,15),
(1,2)(5,8)(6,9)(7,10) ]), isGraph := true,
names := [ 1 .. 16 ], order := 16, representatives := [ 1 ],
schreierVector :=
[ -1, 4, 3, 3, 3, 2, 1, 4, 4, 4, 3, 2, 1, 3, 2, 1 ] )
gap> IsNG(gamma);
true
```

In Appendix 3.A, we also make use of the symbolic algebra system, Maple [8]. In particular, our computations rely heavily on the `Groebner` package. Similarly to the GAP code, we use the following environment to illustrate our computations in Maple.

```
> srg_rel:={mu*(v-k-1)-k*(k-l-1),lambda-mu-r-s,mu-k-r*s};
> ord:=tdeg(t,d,v,k,lambda,mu,r,s);
> G:=Groebner[Basis](srg_rel,ord);
```

1.3 Attribution of content

In this thesis we present new research alongside important background results necessary for an intelligible discussion and justification of our findings. We specify whether the results of each section is background material, attained through a collaborative effort or attained solely by the author.

- Chapter 1: This chapter consists of introductory material.
- Chapter 2: This chapter largely contains background material. Section 2.4.1 appears to be new, although quite simple. Corollary 2.5 is an easy consequence of known results.
- Chapter 3: Unless otherwise stated, this chapter is based on original work of the author.
- Chapter 4: Unless otherwise stated, this chapter is based on original work of the author, including the first part of the collaborative paper [31].
- Chapter 5: The results of Sections 5.1 and 5.2.3 are joint results of Goryainov, Panasenko and the author. The results in Section 5.2.2 were found by Sergey Goryainov after discussions with the author. The Sections 5.2.2 and 5.2.3 form the latter part of the paper [31].
- Chapter 6: Sections 6.1 to 6.3 contain background material. The results of Section 6.4 are original results of the author. Unless otherwise stated, Section 6.5 contains new results found by Goryainov, Panasenko and the author.
- Appendix A: This appendix and the software it describes are based on original work of the author.

Chapter 2

Regular graphs and subgraphs

In this chapter we introduce graphs, with a focus on the class of edge-regular graphs and its subclass of strongly regular graphs. We also establish standard notation concerning induced subgraphs of graphs.

The notion of strongly regular graphs was first introduced by Bose [9], who used them in the analysis of partial geometries and partially balanced incomplete-block designs. Since then, knowledge of strongly regular graphs has has been applied to the study of many other mathematical objects, including codes, Hadamard matrices, Latin squares, and classical groups. Strongly regular graphs continue to be a popular area of study, partly due to their somewhat unpredictable nature. Cameron [7, Chapter 8] conveys this intuition nicely in the quote "Strongly regular graphs stand on the cusp between the random and the highly structured".

As a superclass of strongly regular graphs, edge-regular graphs are typically less structured and harder to classify. However, we can often infer properties of edge-regular graphs which contain specific substructures, and this is the approach often taken in the literature (for example, [23] and [14, Chapter 1]).

We also introduce the block intersection polynomial, a combinatorial tool derived from "counting in two ways". Similar counting arguments have been applied to many objects defined by combinatorial properties, including nets [9], block designs [21], partial geometries [9], strongly regular graphs [11] and edge-regular graphs [23].

Mendelsohn [45] and Cameron and Soicher [17] apply the argument of double counting to designs, and Soicher [53] subsequently extends these results to graphs.

Many of the results in the aforementioned references can be seen as an application of the block intersection polynomial. Our approach will be based on that of Soicher [53]. Similar applications have already appeared in the literature, for example in Soicher [53], Greaves and Soicher [38], and Momihara and Suda [48].

2.1 Graphs

First, we give some definitions and background on the theory of graphs. For an introduction and overview of graph theory, see [59] and [28].

A *graph* is an ordered pair $\Gamma = (V, E)$, where V is a finite set and E is a set of subsets of size 2 of V. Then, the members of V are called the *vertices* of Γ, and the members of E are called the *edges* of Γ. We denote the set of vertices of the graph Γ by $V(\Gamma)$, and the set of edges of Γ by $E(\Gamma)$.

Now let Γ be a graph. The *order* of Γ is the cardinality $|V(\Gamma)|$ of its vertex set, and the *size* of Γ is the cardinality $|E(\Gamma)|$ of its edge set. For any two distinct vertices u, w of Γ, we denote by uw the set $\{u, w\}$, and u, w are said to be *adjacent* if $uw \in E(\Gamma)$. We do not consider a vertex to be adjacent to itself. An edge e is *incident* to a vertex u if e contains u. Similarly, a vertex u is incident to an edge e if u is contained in e.

Let u be a vertex of Γ. The *neighbourhood* of u is the set of vertices adjacent to u, and is denoted by $\Gamma(u)$. The *degree* of u is the cardinality $|\Gamma(u)|$ of its neighbourhood.

The *complement* of Γ, denoted by $\overline{\Gamma}$, is the graph with vertex set $V(\overline{\Gamma}) := V(\Gamma)$, and for distinct vertices $u, w \in V(\Gamma)$, we have $uw \in E(\overline{\Gamma})$ if and only if $uw \notin E(\Gamma)$.

A *path* in Γ is a sequence of distinct vertices $u_0, u_1, \ldots, u_n$, such that $u_i u_{i+1} \in E(\Gamma)$ for all $0 \leq i \leq n-1$. The *length* of a path $u_0, u_1, \ldots, u_n$ is the number n. For vertices $u, w \in \Gamma$, the *distance* between u and w, denoted by $d_\Gamma(u, w)$, is the length of a shortest path between them (by convention, $d(w, w) = 0$ for any vertex w). The graph Γ is *connected* if a path exists between any two distinct vertices. If Γ is connected, the *diameter* of Γ is defined as the maximum distance between any two distinct vertices of Γ.

Let v be the order of Γ. The *adjacency matrix* of Γ, $A(\Gamma)$, is the $v \times v$ matrix indexed by $V(\Gamma)$ such that $A(\Gamma)_{xy} = 1$ if $xy \in E(\Gamma)$, and $A(\Gamma)_{xy} = 0$ otherwise. The *spectrum* of the graph Γ, $\mathrm{Spec}(\Gamma)$, is the multiset of eigenvalues of its adjacency matrix. For α an eigenvalue of Γ, the *multiplicity* of α is the number of times α occurs in the spectrum of Γ. For references on the spectra of graphs, see [24] and [15].

2.2 Edge-regular graphs

A graph Γ is *k-regular* if every vertex of Γ has degree k.

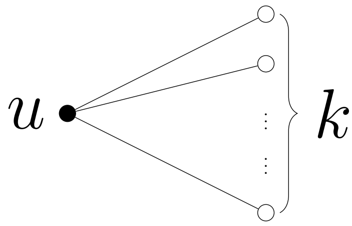

Figure 2.1: A vertex u in a k-regular graph

Two important families of regular graphs are the complete graphs, and the null graphs. A graph Γ is a *complete* graph if for every pair of distinct vertices $u, w \in V(\Gamma)$, we have $uw \in E(\Gamma)$. This graph is denoted by K_v, and is $(v-1)$-regular. A graph Γ is a *null* graph if for every pair of distinct vertices $u, w \in V(\Gamma)$, we have $uw \notin E(\Gamma)$. This graph is denoted by O_v, and is 0-regular.

Figure 2.2: The graphs K_5 and O_5

A graph Γ of order v is *edge-regular* with *parameters* (v, k, λ) if Γ is non-null, k-regular, and every pair of adjacent vertices have exactly λ common neighbours. We denote by $\mathrm{ERG}(v, k, \lambda)$ the set of edge-regular graphs with parameters (v, k, λ).

A graph Γ is *co-edge-regular* with *parameters* (v, k, μ) if Γ is non-complete, k-regular and every pair of distinct non-adjacent vertices have exactly μ common neighbours.

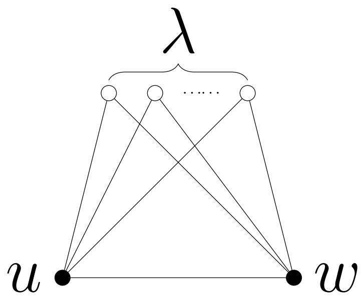

Figure 2.3: Adjacent vertices u, w of an edge-regular graph

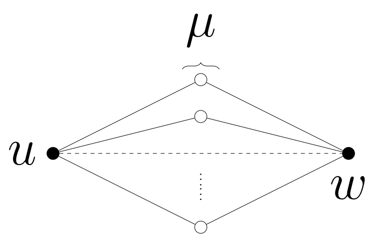

Figure 2.4: Nonadjacent vertices u, w of a co-edge-regular graph

A graph Γ is *amply regular* with *parameters* (v, k, λ, μ) if Γ is edge-regular with parameters (v, k, λ), and every pair of vertices at distance 2 from each other have exactly μ common neighbours.

A graph Γ is *strongly regular* with *parameters* (v, k, λ, μ) if Γ non-complete, edge-regular with parameters (v, k, λ) and every pair of distinct nonadjacent vertices have exactly μ common neighbours. We denote by $\mathrm{SRG}(v, k, \lambda, \mu)$ the set of strongly regular graphs with parameters (v, k, λ, μ).

Note that a strongly regular graph is amply-regular with the same parameters. Further, a graph is strongly regular with parameters (v, k, λ, μ) if and only if it is edge-regular with parameters (v, k, λ) and co-edge-regular with parameters (v, k, μ).

2.3 Induced subgraphs and regular sets

Consider a set of vertices $U \subseteq V(\Gamma)$. The *induced subgraph* of Γ on U, denoted by $\Gamma[U]$, is the graph with vertex set U, and vertices in $\Gamma[U]$ are adjacent if and only if they are adjacent in Γ.

Let U be a subset of vertices of Γ. Then U is an *m-regular set* if every vertex in $V(\Gamma)\backslash U$ is adjacent to the same number $m > 0$ of vertices in U. Furthermore, U is a *(d, m)-regular set* if U is an m-regular set and $\Gamma[U]$ is a d-regular graph.

We comment that the term "regular set" was used by Neumaier in the 1980s [50]. It is Cardoso and Rama [18] who introduce the notation "(d, m)-regular sets". We will need to consider the notions of m-regular sets and (d, m)-regular sets separately. To avoid confusion, we will not use the term regular set without preceding it with a single parameter or a 2-tuple of parameters.

A *clique* in Γ is a set of pairwise adjacent vertices of Γ, and a clique of size s is called an *s-clique*. A clique S in Γ is *m-regular*, for some $m > 0$, if S is an m-regular set. In this case we say that S has *nexus m* and is an *m-regular clique*. Note that for an s-clique $S \subseteq V(\Gamma)$, the induced subgraph $\Gamma[S]$ is a complete graph and if S is m-regular then S is an $(s-1, m)$-regular set.

2.4 Intersection numbers for subgraphs

We will now introduce the results of Soicher [53] involving intersection numbers in graphs, but restrict our attention to the results necessary for this Thesis. First, we present a system of linear equations involving intersection numbers and a polynomial which is used in subsequent chapters. We will then focus on when a set of vertices in a graph is m-regular, finding some relations not found in the available literature.

Let Γ be a graph, and $S \subseteq V(\Gamma)$. The *i-th intersection number* $n_i(\Gamma, S)$ is the number of vertices u in $V(\Gamma) \setminus S$ such that

$$|\Gamma(u) \cap S| = i.$$

For any $T \subseteq V(\Gamma)$, we define

$$\lambda_T(\Gamma, S) = |\{u \in V(\Gamma) \setminus S : T \subseteq \Gamma(u)\}|.$$

Let $s := |S|$. For $0 \le j \le s$, we define

$$\lambda_j(\Gamma, S) = \binom{s}{j}^{-1} \sum_{T \subseteq S, |T|=j} \lambda_T(\Gamma, S).$$

The numbers $\lambda_j(\Gamma, S)$ can be seen as the average number of neighbourhoods of vertices not in S that a subset of size j of S is contained in. The intersection numbers and the numbers $\lambda_j(\Gamma, S)$ satisfy certain integer linear equations.

Theorem 2.1. *Let Γ be a graph, $S \subseteq V(\Gamma)$ and $s = |S|$. For $0 \leq j \leq s$, we have*

$$\binom{s}{j}\lambda_j(\Gamma, S) = \sum_{u \in V(\Gamma)\setminus S} \binom{|S \cap \Gamma(u)|}{j} \tag{2.1}$$

$$= \sum_{i=0}^{s} \binom{i}{j} n_i(\Gamma, S) \tag{2.2}$$

Proof. This is proven by using a simple double counting argument by Soicher [53, Theorem 2.1].

□

The block intersection polynomial is one of the main tools of this Thesis. Soicher [53] finds useful properties of the polynomial which follow from the properties of intersection numbers, like non-negativity.

Let $[\lambda_0, \lambda_1, \lambda_2]$ be a sequence of real numbers and y be a real number, with $y \geq 2$. Then we define the *block intersection polynomial*

$$B(x, y, [\lambda_0, \lambda_1, \lambda_2]) := x(x+1)\lambda_0 - 2xy\lambda_1 + y(y-1)\lambda_2. \tag{2.3}$$

We present some of the properties of the block intersection polynomial, which are proven for a more general version of the block intersection polynomial in Cameron and Soicher [17].

Theorem 2.2. *Let y be a non-negative integer, with $y \geq 2$, let $n_0, ..., n_y$ be non-negative real numbers and $\lambda_0, \lambda_1, \lambda_2$ be real numbers, such that*

$$\binom{y}{j}\lambda_j = \sum_{i=0}^{y} \binom{i}{j} n_i \quad (j = 0, 1, 2) \tag{2.4}$$

and let $B(x)$ be the block intersection polynomial $B(x, y, [\lambda_0, \lambda_1, \lambda_2])$ defined in (2.3).

Then for all integers m, $B(m) \geq 0$. Furthermore, $B(m) = 0$ if and only if $n_i = 0$ for all $i \notin \{m, m+1\}$.

We will use this theorem to investigate a graph Γ with given regularity properties, and a subset $S \subseteq V(\Gamma)$ with given properties. In Section 3.3, this approach is used to derive bounds on the order of a d-regular induced subgraph in a strongly regular graph with given parameters. The next section describes a general framework for our approach, and shows that our computations can be quite efficient.

2.4.1 Calculations with block intersection polynomials

The properties of the block intersection polynomial found in Theorem 2.2 are used throughout this thesis. In our investigations, we often start by assuming we have a graph Γ and set $S \subseteq V(\Gamma)$ of size s. In future chapters, we will see that appropriate assumptions on the structure of Γ and the subgraph $\Gamma[S]$ enables us to determine the values $\lambda_i(\Gamma, S)$ in terms of s and other given parameters, without knowledge of the values $n_j(\Gamma, S)$. We can then determine if $B(m, s, [\lambda_0, \lambda_1, \lambda_2]) \geq 0$ for all integers m. If we find this is not true, by Theorem 2.2 we have proven there does not exist such a subset S.

Therefore, we would like to be able to determine whether $B(m, s, [\lambda_0, \lambda_1, \lambda_2]) \geq 0$ for all integers m efficiently. In the discussion below we give a method for doing so.

Let y and $\lambda_0, \lambda_1, \lambda_2$ be real numbers and $B(x)$ be the block intersection polynomial $B(x, y, [\lambda_0, \lambda_1, \lambda_2])$ defined in (2.3). If $\lambda_0 < 0$, B is a quadratic in x with negative leading coefficient, so there exists an integer m with $B(m) < 0$. If $\lambda_0 = 0$, B has degree at most 1 in x, and is non-negative at all integers if and only if B is a non-negative constant function.

Suppose now $\lambda_0 > 0$. Then B attains its minimum value at

$$x_y = \frac{2y\lambda_1 - \lambda_0}{2\lambda_0}.$$

Now consider $[x_y]$, the value of x_y rounded to a nearest integer (if $x_y + 1/2$ is an integer, we define $[x_y] = x_y + 1/2$). The minimum of B at any integer value is then $B([x_y])$. Therefore, $B(m) \geq 0$ for all integers m if and only if $B([x_y]) \geq 0$.

The most computationally costly calculation in this method is the rounding of x_y to $[x_y]$. If x_y is a rational number, the rounding operation can be done exactly and efficiently. In this thesis, a block intersection polynomial B will always have rational coefficients, so x_y is always a rational number.

It is also interesting to note the following graph theoretical interpretation of x_y. Let Γ be a graph, $S \subseteq V(\Gamma)$ of order s and $\lambda_i = \lambda_i(\Gamma, S)$ for $i = 0, 1, 2$. Then $s\lambda_1$ is the number of edges uw such that $u \in V(\Gamma) \setminus S$ and $w \in S$. Hence $x_s + 1/2$ is the average of $|\Gamma(u) \cap S|$ over all $u \in V(\Gamma) \setminus S$.

2.5 Intersection numbers of m-regular sets

Now we restrict our attention to when we have a m-regular set. In this case we give relations between the numbers λ_j which have not been noted down in the available literature, and which we will find useful in subsequent chapters.

As a simple corollary of Theorem 2.2, we can deduce properties of the related block intersection polynomial from the existence of an m-regular set, and vice-versa.

Corollary 2.3. *Let Γ be a graph of order v, $S \subseteq V(\Gamma)$ and $s = |S|$, with $s \geq 2$. Let $\lambda_j = \lambda_j(\Gamma, S)$ for $j = 0, 1, 2$, and $B(x)$ be the block intersection polynomial $B(x, s, [\lambda_0, \lambda_1, \lambda_2])$.*

Then the set S is an m-regular set if and only if $B(m) = B(m - 1) = 0$

Proof. This easily follows from the last part of Theorem 2.2. $\square$

Let Γ be a graph of order v, $S \subseteq V(\Gamma)$ and $s = |S|$, with $s \geq 2$. We will be interested in the case where S is an m-regular set of Γ.

Consider the right-hand side of the equations (2.1). Then for each $u \in V(\Gamma) \setminus S$, we have $|S \cap \Gamma(u)| = m$. By equation (2.1), we immediately see that

$$\binom{s}{j}\lambda_j = (v - s)\binom{m}{j} \tag{2.5}$$

for every j.

In particular, we can derive a recursive expression for λ_j when we have an m-regular set.

Lemma 2.4. *Let s, t, m be non-negative integers, $s \geq 2$, $2 \leq t \leq s$ and $1 \leq m \leq s$. Further, let λ_j be real numbers for each $0 \leq j \leq t$. Then*

$$\binom{s}{j}\lambda_j = (v-s)\binom{m}{j}. \tag{2.6}$$

for all $j \leq t$ if and only if

$$(s-j)\lambda_{j+1} = (m-j)\lambda_j \tag{2.7}$$

for all $j < t$.

Proof. Suppose the equations (2.6) hold. In particular, we have

$$\binom{s}{j+1}\lambda_{j+1} = (v-s)\binom{m}{j+1}, \tag{2.8}$$

$$\binom{s}{j}\lambda_j = (v-s)\binom{m}{j}. \tag{2.9}$$

for each $j < t$.

First we consider the case $j < m$. Then we also have $j < s$ and $\lambda_{j+1} \neq 0$. Multiply both sides of (2.8) by $(j+1)!$ and (2.9) by $j!$, and we find that

$$s(s-1)\dots(s-j)\lambda_{j+1} = (v-s)m(m-1)\dots(m-j) \tag{2.10}$$

$$s(s-1)\dots(s-j+1)\lambda_j = (v-s)m(m-1)\dots(m-j+1) \tag{2.11}$$

Dividing equation (2.10) by (2.11) and multiplying both sides by λ_j, we prove the equality (2.7) holds.

Now consider the case $j = m$. By definition, $\lambda_{j+1} = 0$, and the equality (2.7) holds because $m - j = 0$.

Lastly, consider the case $j > m$. By induction or otherwise, we see that $\lambda_{j+1} = \lambda_j = 0$, so the equality holds.

Suppose the equality (2.7) holds for all $0 \leq j < t$. We will prove that the equality (2.6) holds by induction on j. By definition, $\lambda_0 = v - s$, and by assumption, we have $s\lambda_1 = m\lambda_0$.

Now assume equality (2.6) holds for all values $j < i$. First we consider the case

$i < m$. By the assumption of the proposition and the inductive assumption, we have

$$\lambda_i = \left(\frac{s-i}{m-i}\right)\lambda_{i+1}$$
$$\binom{s}{i}\lambda_t = (v-s)\binom{m}{i}.$$

By substituting for the value of λ_i and dividing by $i+1$, we have shown the equality (2.6) holds for $j = i$.

The case $i \geq m$ is easily verified as equation (2.7) implies that $\lambda_{m+1} = \cdots = \lambda_t = 0$.

By induction, we have shown that (2.6) holds for each j. □

Corollary 2.3 tells us that we can determine whether the set S is m-regular from λ_0, λ_1 and λ_2 using the block intersection polynomial. This can be observed in the following result, in which we collect three other conditions that are equivalent to the set S being m-regular.

Corollary 2.5. *Let Γ be a graph of order v, $S \subseteq V(\Gamma)$ and $s = |S|$, with $2 \leq s < v$. Let $\lambda_j = \lambda_j(\Gamma, S)$ for $0 \leq j \leq s$, let m be a positive integer, and let $B(x)$ be the block intersection polynomial $B(x, s, [\lambda_0, \lambda_1, \lambda_2])$.*

The following are equivalent:

1. *S is an m-regular set.*

2. *$B(m) = B(m-1) = 0$.*

3. $$\binom{s}{j}\lambda_j = (v-s)\binom{m}{j}.$$

   for each $j = 0, 1, 2$.

4. $$(s-j)\lambda_{j+1} = (m-j)\lambda_j$$

   for each $j = 0, 1$.

Proof. (1. $\iff$ 2.) This is Corollary 2.3.

(1. $\implies$ 3.) This is Equation (2.5).

(3. $\iff$ 4.) This follows from Lemma 2.4, where $t = 2$.

(3. $\implies$ 2.) We have

$$\begin{aligned}\lambda_0 &= v - s,\\ \lambda_1 &= (v-s)\frac{m}{s},\\ \lambda_2 &= (v-s)\frac{m(m-1)}{s(s-1)}.\end{aligned}$$

Using the above equalities, we have

$$\begin{aligned}B(m) &= m(m+1)(v-s) - 2ms(v-s)\frac{m}{s} + (v-s)\frac{m(m-1)}{s(s-1)}\\ &= (v-s)(m(m+1) - 2m + m(m-1))\\ &= 0,\\ B(m-1) &= m(m-1)(v-s) - 2(m-1)s(v-s)\frac{m}{s} + (v-s)\frac{m(m-1)}{s(s-1)}\\ &= (v-s)(m(m-1) - 2m(m-1) + m(m-1))\\ &= 0.\end{aligned}$$

□

We will be mostly interested in the recursive expressions with $j = 0, 1, 2$. We finish this section by displaying λ_0 and the expressions in (2.7) for $j = 0, 1$.

$$\lambda_0 = v - s. \tag{2.12}$$

$$s\lambda_1 = m\lambda_0 \tag{2.13}$$

$$(s-1)\lambda_2 = (m-1)\lambda_1 \tag{2.14}$$

As noted in Section 2.4.1, we will often be able to determine the values $\lambda_i(\Gamma, S)$ without knowledge of the values $n_j(\Gamma, S)$ by assuming additional structure on Γ and the subgraph $\Gamma[S]$. The equations (2.12),(2.13) and (2.14) can then reveal new relations between the assumed parameters of the graph Γ and the values m and s, where $S \subseteq V(\Gamma)$ is an m-regular set of size s. In particular, this is used in Lemma 4.5

and Lemma 6.4 to restrict the possible parameters of a graph containing a m-regular s-clique or (d, m)-regular set, respectively.

Chapter 3

Bounds for regular induced subgraphs of strongly regular graphs

The question of finding the maximum order of a d-regular induced subgraph of a given graph Γ is a generalisation of many problems in graph theory. Some examples of these include finding the independence number, clique number and the order of a maximum induced matching in a given graph. In general, finding a d-regular induced subgraph of a given graph Γ is computationally hard (see Asahiro et al. [3]). Significant improvements in computational time can be made by using bounds on the order of a d-regular induced subgraph of Γ to reduce the search space of the problem.

Haemers [40] gives an upper and lower bound on the order of a d-regular induced subgraph of a v-vertex k-regular graph with given least and second largest eigenvalues, which generalises an unpublished result of Hoffman. More recently, Cardoso, Karminski and Lozin [19] derive the same upper bound as a consequence of semidefinite programming methods which can be applied to any graph. Considering a strongly regular graph Γ with parameters (v, k, λ, μ), Neumaier [50] derives the same upper and lower bounds on the order of a d-regular induced subgraph of Γ, through the use of a combinatorial argument.

Greaves and Soicher [38] analyse an upper bound on the order of cliques in an edge-regular graph with given parameters, called the clique adjacency bound. They prove that given any strongly regular graph Γ, the clique adjacency bound is at least as good as the well-known Delsarte bound. Furthermore, they find many strongly regular graphs for which the clique adjacency bound is strictly better than the Delsarte bound.

In this chapter, we generalise certain results of Greaves and Soicher [38], where instead of cliques, we will consider d-regular induced subgraphs. First, we present known results on the spectra of strongly regular graphs and bounds on regular induced subgraphs. Given feasible strongly regular graph parameters (v, k, λ, μ) and a non-negative integer d, we use the block intersection polynomials defined in Chapter 2 to determine upper and lower bounds on the order of a d-regular induced subgraph of any strongly regular graph with parameters (v, k, λ, μ). Our new bounds are at least as good as the bounds on the order of a d-regular induced subgraph of a k-regular graph determined by Haemers [40]. We carry out computations using the AGT package to verify the new bounds beat Haemers' bounds relatively often for strongly regular graphs of small order. Further, we give a sufficient condition that, if true, would show that for each non-negative integer d, our new upper bound improves on Haemers' upper bound for infinitely many strongly regular graphs.

3.1 Spectra of strongly regular graphs

For a strongly regular graph Γ with parameters (v, k, λ, μ), it is known that Γ has at most 3 distinct eigenvalues, with largest eigenvalue k (see Brouwer and Haemers [15, Theorem 9.1.2]). The *restricted eigenvalues* of a strongly regular graph are the eigenvalues of the graph with eigenspaces perpendicular to the all-ones vector. We often denote these eigenvalues by ρ, σ, with $k \geq \rho > \sigma$. The following shows that the eigenvalues of strongly regular graphs only depend on the parameters of the graph.

Proposition 3.1. *Let Γ be in $SRG(v, k, \lambda, \mu)$ and $\rho > \sigma$ be the restricted eigenvalues of Γ. Then*

1. *ρ and σ are uniquely determined from the parameters (v, k, λ, μ), we have*

$\rho \geq 0, \sigma < 0$, and the following relations hold.

$$\begin{aligned}
\mu(v-k-1) &= k(k-\lambda-1) \\
\lambda - \mu &= \rho + \sigma \\
\mu - k &= \rho\sigma
\end{aligned}$$

2. *If ρ, σ are not integers, then there exists a positive integer n such that $(v, k, \lambda, \mu) = (4n+1, 2n, n-1, n)$.*

Proof. This is a routine calculation that uses the properties of the adjacency matrix of a strongly regular graph, and can be found in Brouwer and Haemers [15, Theorem 9.1.3]. □

Using this Proposition, we can now derive the following useful identity.

Lemma 3.2. *Let Γ be in SRG(v, k, λ, μ) with restricted eigenvalues $\rho > \sigma$. Then $v\mu = (k-\sigma)(k-\rho)$.*

Proof. This follows immediately from Proposition 3.1. □

3.2 Spectral bounds

In his thesis, Haemers [40] derives bounds on the order of induced subgraphs of regular graphs using eigenvalue techniques. We present these bounds and emphasise the cases involving (d, m)-regular sets.

Proposition 3.3. *Let Γ be a k-regular graph of order v with smallest eigenvalue σ. Suppose Γ has an induced subgraph Δ of order $y > 0$, and average vertex-degree d. Then*

$$y \leq v\left(\frac{d-\sigma}{k-\sigma}\right).$$

Furthermore, $y = v(d-\sigma)/(k-\sigma)$ if and only if $V(\Delta)$ is a $(d, d-\sigma)$-regular set.

Proof. This is a standard result that uses eigenvalue interlacing. The proof can be found in Haemers [40, Theorem 2.1.4]. □

A lower bound can also be derived when considering connected graphs. Note that this bound need not be positive.

Proposition 3.4. *Let Γ be a k-regular graph of order v with second largest eigenvalue ρ. Suppose Γ has an induced subgraph Δ of order $y > 0$, and average vertex-degree d. Then*

$$y \geq v\left(\frac{d-\rho}{k-\rho}\right).$$

Furthermore, $y = v(d-\rho)/(k-\rho)$ if and only if $V(\Delta)$ is a $(d, d-\rho)$-regular set.

Proof. This is a standard result that uses eigenvalue interlacing. The proof can be found in Haemers [40, Theorem 2.1.4]. □

For a k-regular graph Γ of order v and with smallest eigenvalue σ, we define the upper bound of Haemers

$$\mathrm{Haem}_{\geq}(\Gamma, d) := v\left(\frac{d-\sigma}{k-\sigma}\right) \tag{3.1}$$

and if Γ is connected with second largest eigenvalue ρ, we define the lower bound of Haemers

$$\mathrm{Haem}_{\leq}(\Gamma, d) := v\left(\frac{d-\rho}{k-\rho}\right). \tag{3.2}$$

We note that $\mathrm{Haem}_{\geq}(\Gamma, 0)$ coincides with the well-known Hoffman bound (also known as the ratio bound), so the bounds of Haemers' generalise the Hoffman bound.

Example 3.5. In this example, we will see that for certain cases of (strongly) regular graphs, the upper and lower bounds of Haemers are attained.

For $n \geq 2$, the *square lattice graph* $L_2(n)$ has vertex set $\{1, 2, ..., n\} \times \{1, 2, ..., n\}$, and two distinct vertices are joined by an edge precisely when they have the same value at one coordinate. This graph is strongly regular with parameters $(n^2, 2(n-1), n-2, 2)$, and has eigenvalues $k = 2n-2, \rho = n-2, \sigma = -2$.

Now consider the induced subgraph Δ with vertex set consisting of the complement of two distinct columns. Then Δ is a $(2n-4)$-regular induced subgraph of order $n^2 - 2n$, which is the lower bound of Haemers.

Now consider the induced subgraph Δ with vertex set consisting of the complement of a maximum-size independent set. Then Δ is a $(2n-4)$-regular induced

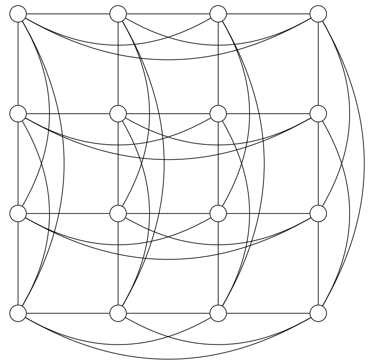

Figure 3.1: The Square lattice graph $L_2(4)$.

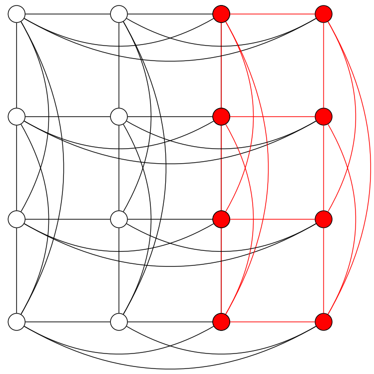

Figure 3.2: A regular induced subgraph attaining Haemers' lower bound

subgraph of order $n^2 - n$, which is the upper bound of Haemers.

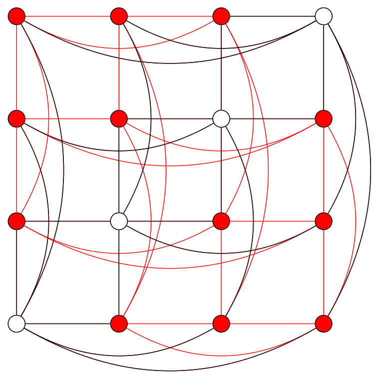

Figure 3.3: A regular induced subgraph attaining Haemers' upper bound

$\triangle$

3.3 The Regular Adjacency Bounds

Our aim for the rest of the chapter is to generalise some results of Greaves and Soicher [38] to d-regular subgraphs of strongly regular graphs. First we derive bounds on the order of a d-regular induced subgraph of a strongly regular graph, using the block intersection polynomial.

Let $\Gamma \in \mathrm{SRG}(v,k,\lambda,\mu)$. Suppose Δ is a d-regular induced subgraph of order $y \geq 2$ of Γ. Let $S = V(\Delta)$, and consider $\lambda_i = \lambda_i(\Gamma, S)$ for $i = 0,1,2$.

By definition, $\lambda_0 = v - y$. As Γ is k-regular, a vertex $u \in S$ has exactly $k-d$ neighbours in $V(\Gamma) \setminus S$. Therefore, for each $u \in S$, $\lambda_{\{u\}} = k - d$, and so $\lambda_1 = k - d$.

Now consider $\binom{y}{2}\lambda_2$. Note that this is the number of paths of length 2 with distinct end points in S, and midpoint in $V(\Gamma) \setminus S$. Any subset $\{u,w\} \subseteq S$ lies in one of the following sets:

$$E = \{\{u,w\} \subseteq S : u, w \text{ are adjacent in } \Gamma\}.$$
$$F = \{\{u,w\} \subseteq S : u, w \text{ are non-adjacent in } \Gamma\}.$$

Let $e = |E|, f = |F|$, and consider any pair of distinct vertices $u, w \in S$.

1. If $\{u,w\} \in E$, there are exactly λ paths of length 2 between them in Γ.

2. If $\{u,w\} \in F$, there are exactly μ paths of length 2 between them in Γ.

Therefore, there are a total of $e\lambda + f\mu$ paths of length 2 between distinct vertices in S. We can also count the number of paths of length 2 between distinct vertices in S where the midpoint also lies in S. This is exactly

$$\sum_{u \in S} \binom{|\Delta(u)|}{2} = y\binom{d}{2}.$$

From this we deduce that

$$\binom{y}{2}\lambda_2 = e\lambda + f\mu - y\binom{d}{2}.$$

We also know that

$$e = \frac{yd}{2}, e + f = \binom{y}{2},$$

so we can eliminate e, f and find that

$$2\binom{y}{2}\lambda_2 = (\lambda - \mu + 1)yd + y(y-1)\mu - yd^2.$$

We now define the *regular adjacency polynomial* for the graph Γ, or *rap*, as the block intersection polynomial

$$\begin{aligned} R_\Gamma(x, y, d) :&= B(x, y, [\lambda_0, \lambda_1, \lambda_2]) \\ &= x(x+1)(v-y) - 2xyk + (2x + \lambda - \mu + 1)yd + y(y-1)\mu - yd^2. \end{aligned}$$

This is the polynomial found in Soicher [54, Theorem 1.2], applied with constant degree sequence $(d, d, ..., d)$. Note that we are dealing with strongly regular graphs, so we do not need to consider the diameter condition stated in the theorem.

This polynomial has some useful properties, which come from the fact that it is a block intersection polynomial.

Theorem 3.6. *Let Γ be in SRG(v, k, λ, μ) and Δ be a d-regular induced subgraph of order $y \geq 2$ in Γ. Then $R_\Gamma(m, y, d) \geq 0$ for all integers m. Moreover, $R_\Gamma(m, y, d) = R_\Gamma(m-1, y, d) = 0$ for some integer m if and only if $V(\Delta)$ is a (d, m)-regular set.*

Proof. This is an application of Theorem 2.2, where we use the above λ_i in equation (2.3) to define our block intersection polynomial. $\square$

Note that given a set $S \subseteq V(\Gamma)$ such that $\Gamma[S]$ is d-regular, it is not necessarily true that a proper subset of S induces a d-regular subgraph. Because of this, how we define bounds from the properties of the regular adjacency polynomial will be slightly different to how the clique adjacency bound is defined from the properties of the clique adjacency polynomial.

Consider the set

$$S_d := \{y \in \{d+1, \dots, v\} : \text{ for all integers } x, R_\Gamma(x, y, d) \geq 0\}.$$

We define the *regular adjacency upper bound*, or *raub* of the strongly regular graph

Γ as

$$\mathrm{Rab}_{\geq}(\Gamma, d) := \begin{cases} \max(S_d) & S_d \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and we define the *regular adjacency lower bound*, or *ralb* of the strongly regular graph Γ as

$$\mathrm{Rab}_{\leq}(\Gamma, d) := \begin{cases} \min(S_d) & S_d \neq \emptyset \\ v+1 & \text{otherwise.} \end{cases}$$

Note that these bounds are the same for any two distinct graphs in $\mathrm{SRG}(v, k, \lambda, \mu)$. We also comment that by Section 2.4.1, there are at most $(v-d)$ calculations needed to compute one (or both) of these bounds.

After dealing with a trivial case, we can now use Theorem 3.6 to prove that the graph Γ has no non-empty d-regular induced subgraph of order greater than $\mathrm{Rab}_{\geq}(\Gamma, d)$ or less than $\mathrm{Rab}_{\leq}(\Gamma, d)$.

Theorem 3.7. *Let Γ be in $SRG(v, k, \lambda, \mu)$ and Δ be a d-regular induced subgraph of order $y > 0$ in Γ. Then*

$$Rab_{\leq}(\Gamma, d) \leq y \leq Rab_{\geq}(\Gamma, d).$$

Proof. It is easy to see that Δ has to have at least $d+1$ vertices. If $y \geq 2$, then by Theorem 3.6, we have $R_\Gamma(x, y, d) \geq 0$ for all integers x. By the definitions of the raub and ralb,

$$\mathrm{Rab}_{\leq}(\Gamma, d) \leq y \leq \mathrm{Rab}_{\geq}(\Gamma, d).$$

The only case left to consider is when $y = 1$. As Δ has at least $d+1$ vertices, we must have $d = 0$. Consider

$$R_\Gamma(x, 1, 0) = x\left((x+1)(v-1) - 2k\right).$$

This polynomial in x has roots $x_1 = 0$ and $x_2 = 2k/(v-1) - 1$. As $k \leq v-1$, we have $1 \geq x_2 \geq -1$.

Therefore, we have the following three cases.

1. $-1 \leq x_2 < 0$ and $R(x, 1, 0)$ is negative only for x lying in an open interval contained in $(-1, 0)$.

2. $x_2 = 0$ and $R(x, 1, 0)$ is non-negative for all x.

3. $0 < x_2 \leq 1$ and $R(x, 1, 0)$ is negative only for x lying in an open interval contained in $(0, 1)$.

In each case, for all integers x we have $R(x, 1, 0) \geq 0$. By definition, we see that

$$\mathrm{Rab}_{\leq}(\Gamma, 0) \leq 1 \leq \mathrm{Rab}_{\geq}(\Gamma, 0).$$

$\square$

3.4 Comparison of bounds

We will now compare the bounds of Haemers from Section 3.2 with the raub and ralb defined in Section 3.3. For $\Gamma \in \mathrm{SRG}(v, k, \lambda, \mu)$, the next three propositions show that $\mathrm{Rab}_{\geq}(\Gamma, d) \leq \lfloor \mathrm{Haem}_{\geq}(\Gamma, d) \rfloor$, and $\mathrm{Rab}_{\leq}(\Gamma, d) \geq \lceil \mathrm{Haem}_{\leq}(\Gamma, d) \rceil$.

We fix strongly regular graph parameters (v, k, λ, μ) and the corresponding restricted eigenvalues ρ, σ where $\rho > \sigma$. Note that the upper and lower bounds $\mathrm{Haem}_{\geq}(\Gamma, d), \mathrm{Haem}_{\leq}(\Gamma, d)$ on d-regular induced subgraphs derived in Section 3.2 are the same for all graphs Γ in $\mathrm{SRG}(v, k, \lambda, \mu)$.

The approach we take relies on the following useful observations. At most values of y and d, the polynomial $R_\Gamma(x, y, d)$ is a quadratic in x with positive leading coefficient. For any fixed value of y in the ranges $0 < y < \mathrm{Haem}_{\leq}(\Gamma, d)$ and $\mathrm{Haem}_{\geq}(\Gamma, d) < y < v$, we will see that this quadratic in x is negative on an open interval of length strictly greater than 1. Every interval of length more than 1 must contain an integer, with which we can then use in applying Theorem 3.6.

We restrict to values of y that are strictly less than v and take cases on μ.

Proposition 3.8. *Let Γ be in $SRG(v, k, \lambda, \mu)$ where $\mu \neq 0$ and let d be a non-negative integer of size at most k. For all y such that $0 < y < Haem_{\leq}(\Gamma, d)$ or $Haem_{\geq}(\Gamma, d) < y < v$, there is an integer b_y such that $R_\Gamma(b_y, y, d) < 0$.*

Proof. Let $0 < y < v$. Then $R_\Gamma(x, y, d)$ is a quadratic polynomial with positive leading coefficient. We denote by x_y the point at which $R_\Gamma(x, y, d)$ attains its minimum value in x. If $R_\Gamma(x_y + 1/2, y, d) < 0$, by symmetry of the quadratic around $x = x_y$, $R_\Gamma(x_y - 1/2, y) < 0$, and so we have $R_\Gamma(x, y, d) < 0$ for all $x \in [x_y - 1/2, x_y + 1/2]$.

This is an interval of size 1, so must contain an integer b_y, and $R_\Gamma(b_y, y, d) < 0$. We claim $R(x_y + 1/2, y, d) < 0$ for $y > \text{Haem}_\geq(\Gamma, d)$ or $y < \text{Haem}_\leq(\Gamma, d)$, which proves the result.

For our fixed y, d, the minimum of the quadratic $R_\Gamma(x, y, d)$ is attained at x_y, where

$$\begin{aligned} x_y &= \frac{(2k-2d+1)y - v}{2(v-y)} \\ x_y + \frac{1}{2} &= \frac{(k-d)y}{v-y} \end{aligned}$$

Let $\rho > \sigma$ be the restricted eigenvalues corresponding to the strongly regular graphs parameters (v, k, λ, μ). We then establish the following identity, using the relations for strongly regular graph parameters (and is verified using Maple in Appendix 3.A).

$$\begin{aligned} -\frac{(v-y)}{y} R_\Gamma(x_y + 1/2, y, d) &= \mu y^2 - ((d-\rho)(k-\sigma) + (d-\sigma)(k-\rho))y \\ &\quad + v(d-\sigma)(d-\rho). \end{aligned}$$

Then multiply by μ and deduce the following identity by using Proposition 3.2.

$$-\frac{(v-y)}{y} \mu R_\Gamma(x_y + 1/2, y, d) = (\mu y - (d-\rho)(k-\sigma))(\mu y - (d-\sigma)(k-\rho)) \quad (3.3)$$

(this is also verified using Maple in Appendix 3.A). Consider the right hand side of Equation (3.3) as a quadratic in y. Take the roots of this quadratic,

$$\alpha = \frac{(d-\rho)(k-\sigma)}{\mu}, \beta = \frac{(d-\sigma)(k-\rho)}{\mu}.$$

As $(k-\rho)(d-\sigma) - (k-\sigma)(d-\rho) = (\rho-\sigma)(k-d)$ is positive, we have $\beta \geq \alpha$. By Proposition 3.2, $\beta = \text{Haem}_\geq(\Gamma, d)$ and $\alpha = \text{Haem}_\leq(\Gamma, d)$. We know that $\mu(v-y)/y > 0$, so we have $R_\Gamma(x_y + 1/2, y, d) < 0$ if and only if the right hand side of Equation (3.3) is positive. This is exactly when $y > \beta = \text{Haem}_\geq(\Gamma, d)$ or $y < \alpha = \text{Haem}_\leq(\Gamma, d)$. □

Now we deal with the case when $\mu = 0$.

Proposition 3.9. *Let Γ be in SRG(v, k, λ, μ) where $\mu = 0$ and let d be a non-*

negative integer of size at most k. For all y such that $Haem_{\geq}(\Gamma, d) < y < v$, there is an integer b_y such that $R_\Gamma(b_y, y, d) < 0$.

Proof. For $\mu = 0$, we have $k = \rho, \sigma = -1$. Using the same notation and approach as Proposition 3.8 and using Proposition 3.2, we see that

$$-\frac{(v-y)}{y} R_\Gamma(x_y + 1/2, y, d) = (k-d)(k+1)y - v(k-d)(d+1).$$

The right hand side is strictly greater than 0 for $\mathrm{Haem}_{\geq}(\Gamma, d) < y < v$. Therefore $R(x_y + 1/2, y, d) < 0$ for all such y. Applying a similar argument to Proposition 3.8, we are done. □

Finally, we deal with the case when $y = v$. We would like our bound to allow for a regular subgraph of order v if and only if $d = k$, as this is the degree of the only regular subgraph of order v. The following shows that this is true.

Proposition 3.10. *Let Γ be in $SRG(v, k, \lambda, \mu)$ and let d be a non-negative integer of size at most k. Then there is an integer b_v such that $R_\Gamma(b_v, v, d) < 0$ if and only if $d \neq k$.*

Proof. In this case, $R_\Gamma(x, v, d)$ is a linear function in x. If $k \neq d$, $R_\Gamma(x, y, d)$ is non-constant, so trivially there is such a b_v. Otherwise

$$R_\Gamma(x, v, d) = v((r + s + 1)k + v\mu - \mu - k^2)$$

which is 0 by Proposition 3.2 (and verified by using Maple in Appendix 3.A). □

With the above results we have covered all possible cases needed to prove the following theorem.

Theorem 3.11. *Let Γ be in $SRG(v, k, \lambda, \mu)$ and d be a non-negative integer of size at most k. Then $Rab_{\geq}(\Gamma, d) \leq \lfloor Haem_{\geq}(\Gamma, d) \rfloor$, and $Rab_{\leq}(\Gamma, d) \geq \lceil Haem_{\leq}(\Gamma, d) \rceil$.*

Proof. Note that $\mathrm{Rab}_{\geq}(\Gamma, d), \mathrm{Rab}_{\leq}(\Gamma, d)$ are integers by definition, so we only need to show $\mathrm{Rab}_{\geq}(\Gamma, d) \leq \mathrm{Haem}_{\geq}(\Gamma, d)$ and $\mathrm{Rab}_{\leq}(\Gamma, d) \leq \mathrm{Haem}_{\leq}(\Gamma, d)$.

Take any integer i such that $\mathrm{Haem}_{\geq}(\Gamma, d) < i < v$, or $0 < i < \mathrm{Haem}_{\leq}(\Gamma, d)$ if $\mathrm{Haem}_{\leq}(\Gamma, d) > 0$. Then by the above propositions, $b_i \in \mathbb{Z}$, with $R_\Gamma(b_i, i, d) < 0$.

Thus by Theorem 3.7 and the definitions of $\mathrm{Rab}_{\geq}(\Gamma, d)$ and $\mathrm{Rab}_{\leq}(\Gamma, d)$, the result follows. □

3.4.1 Computational comparison

Now we investigate when the raub and ralb are strictly better than the bounds of Haemers. For this, we can use the AGT package for GAP (see Appendix A).

Let (v, k, λ, μ) be feasible strongly regular graph parameters (see (A.2.9) for the definition of feasible parameters). For any graph $\Gamma \in \mathrm{SRG}(v, k, \lambda, \mu)$ and any non-negative integer $d \leq k$, we can compare the bounds of Haemers and our new bounds by using the following functions:

- `HaemersRegularUpperBound` (A.4.3) to calculate $\mathrm{Haem}_{\geq}(\Gamma, \mathtt{d})$.
- `HaemersRegularLowerBound` (A.4.4) to calculate $\mathrm{Haem}_{\leq}(\Gamma, \mathtt{d})$.
- `RegularAdjacencyUpperBound` (A.4.8) to calculate $\mathrm{Rab}_{\geq}(\Gamma, \mathtt{d})$.
- `RegularAdjacencyLowerBound` (A.4.9) to calculate $\mathrm{Rab}_{\leq}(\Gamma, \mathtt{d})$.

Much of the functionality available in the AGT package can be used to experiment with strongly regular graphs and their parameters. The variable `AGT_Brouwer_Parameters` (A.5.11) contains some of the information about primitive strongly regular graph parameter tuples collected in Brouwer's lists [12] (see (A.5.4) for the definition of primitivity). In particular, for any primitive strongly regular graph parameter tuple (v, k, λ, μ) for which $v \leq 1300$, we can use `AGT_Brouwer_Parameters` to verify if it has been proven that a strongly regular graph with these parameters does not exist.

There are natural trivial bounds to consider in our comparisons of our new bounds with Haemers' bounds. A trivial upper bound on the order of any induced subgraph of a graph of order v is v, and a lower bound on the order of a d-regular induced subgraph is $d+1$. For a graph $\Gamma \in \mathrm{SRG}(v, k, \lambda, \mu)$ and non-negative integer $d \leq k$, we define

$$\mathrm{Haem}^*_{\geq}(\Gamma, d) = \min(\mathrm{Haem}_{\geq}(\Gamma, d), v),$$
$$\mathrm{Haem}^*_{\leq}(\Gamma, d) = \max(\mathrm{Haem}_{\leq}(\Gamma, d), d+1).$$

For each primitive strongly regular parameter tuple (v, k, λ, μ) in `AGT_Brouwer_Parameters` and every non-negative integer d, where $d \leq k$, we will consider a graph $\Gamma \in \mathrm{SRG}(v, k, \lambda, \mu)$, and compare $\mathrm{Rab}_{\geq}(\Gamma, d)$ with $\mathrm{Haem}^*_{\geq}(\Gamma, d)$, and the $\mathrm{Rab}_{\leq}(\Gamma, d)$ with $\mathrm{Haem}^*_{\leq}(\Gamma, d)$. There are currently 1460827 combinations of parameter tuples and d satisfying all the above conditions.

The raub and Haemers' upper bound

The results of our calculations show that $\mathrm{Rab}_{\geq}(\Gamma, d) < \mathrm{Haem}^*_{\geq}(\Gamma, d)$ in 16075 cases, and in 10479 of these cases the regular adjacency polynomial proves there is no possible order for a d-regular induced subgraph in Γ. Out of the remaining 5596 cases, there are 104 cases for which $\mathrm{Rab}_{\geq}(\Gamma, d) < \mathrm{Haem}^*_{\geq}(\Gamma, d) - 1$.

Table 3.1 has first column consisting of all values of d for which we have found a parameter tuple (v, k, λ, μ) such that for every $\Gamma \in \mathrm{SRG}(v, k, \lambda, \mu)$,

$$\mathrm{Rab}_{\geq}(\Gamma, d) < \mathrm{Haem}^*_{\geq}(\Gamma, d) - 1.$$

For each of these d, the second column gives how many parameter tuples this occurs for (Σ_d). The third column give the largest difference $\mathrm{Haem}^*_{\geq}(\Gamma, d) - \mathrm{Rab}_{\geq}(\Gamma, d)$ found for this value of d (Ω_d).

d	Σ_d	Ω_d
0	45	5
1	25	4
2	18	4
3	8	3
4	4	3
5	3	3
6	2	2
7	1	2
8	1	2
9	1	2

Table 3.1: Cases for which $\mathrm{Rab}_{\geq}(\Gamma, d) < \mathrm{Haem}^*_{\geq}(\Gamma, d) - 1$.

The ralb and Haemers' lower bound

The results of our calculations show that $\mathrm{Rab}_{\leq}(\Gamma, d) > \mathrm{Haem}^*_{\leq}(\Gamma, d)$ in 13719 cases, and in 10479 of these cases the regular adjacency polynomial proves there is no possible order for a d-regular induced subgraph in Γ. Out of the 3240 other cases, there are 724 cases for which $\mathrm{Rab}_{\leq}(\Gamma, d) > \mathrm{Haem}^*_{\leq}(\Gamma, d) + 1$.

Table 3.2 has first column consisting of all values of d for which we have found a parameter tuple (v, k, λ, μ) such that for every $\Gamma \in \mathrm{SRG}(v, k, \lambda, \mu)$,

$$\mathrm{Rab}_{\leq}(\Gamma, d) > \mathrm{Haem}^*_{\leq}(\Gamma, d) + 1.$$

For each of these d, the second column gives how many parameter tuples this occurs for (Σ_d). The third column give the largest difference $\mathrm{Rab}_{\leq}(\Gamma, d) - \mathrm{Haem}^*_{\leq}(\Gamma, d)$ found for this value of d (Ω_d).

d	Σ_d	Ω_d
2	23	120
3	29	94
4	43	69
5	49	45
6	76	66
7	75	43
8	69	33
9	71	16
10	61	14
11	60	9
12	34	17
13	36	11
14	27	6
15	18	8
16	16	7
17	15	8
18	8	5
19	8	4
20	3	3
21	2	2
22	1	4

Table 3.2: Cases for which $\mathrm{Rab}_{\leq}(\Gamma, d) > \mathrm{Haem}^*_{\leq}(\Gamma, d) + 1$.

Making small improvements

In many cases we can further improve a bound on the order of a d-regular induced subgraph by considering a well-known divisibility condition, that for any d-regular graph of order y, we must have that 2 divides yd.

Example 3.12. Consider a graph Γ from $\mathrm{SRG}(41, 20, 9, 10)$ and $d = 1$. We can use the AGT package to find that $\mathrm{Rab}_{\geq}(\Gamma, 1) = 7$. But a 1-regular induced subgraph must have even order. Therefore, any 1-regular induced subgraph of Γ has order at most 6. $\triangle$

3.4.2 Improving on Haemers' upper bound

In this section we derive some sufficient conditions for the raub to be strictly better than Haemers' upper bound. To do this, we will use knowledge of type I strongly regular graphs.

A strongly regular graph Γ is of *type I*, or a *conference graph*, if Γ is in $\mathrm{SRG}(4n+1, 2n, n-1, n)$ for some positive integer n. The eigenvalues of a type I strongly regular graph can be calculated as follows.

Proposition 3.13. *Let Γ be in $SRG(v, k, \lambda, \mu)$ and of type I. Then Γ has eigenvalues k, $\rho = (\sqrt{v} - 1)/2$ and $\sigma = (-\sqrt{v} - 1)/2$.*

Let q be a power of a prime, with $q \equiv 1 \pmod 4$. Then the *Paley graph* of order q has vertex set $V = \mathbb{F}_q$, with two vertices adjacent if and only if their difference is a square in $\mathbb{F}_q^*$. Paley graphs are an example of an infinite family of type I strongly regular graphs (see Godsil [34]), and the Paley graph of order q belongs to $\mathrm{SRG}(q, (q-1)/2, (q-5)/4, (q-1)/4)$.

Let Γ be in $\mathrm{SRG}(4n+1, 2n, n-1, n)$ for some positive integer n, and let d be a non-negative integer where $d \leq 2n$. Let ρ, σ be the restricted eigenvalues of Γ, which can be expressed in terms of n by using Proposition 3.13. We now derive some sufficient conditions for $\mathrm{Rab}_{\geq}(\Gamma, d)$ to be strictly less than $\mathrm{Haem}_{\geq}(\Gamma, d)$, by considering the properties of type I graphs. We then show that if a result in analytic number theory holds true, then these conditions can be applied to infinitely many Paley graphs. This would show that for infinitely many strongly regular graphs Γ, we have $\mathrm{Rab}_{\geq}(\Gamma, d) < \mathrm{Haem}_{\geq}(\Gamma, d)$.

First we relate fractional parts of $\mathrm{Haem}_{\geq}(\Gamma, d)$ and a well chosen x coordinate, and then prove that $R(\lfloor x \rfloor, \lfloor \mathrm{Haem}_{\geq}(\Gamma, d) \rfloor)$ is negative. We define $\mathrm{frac}(x) := x - \lfloor x \rfloor$ for $x \in \mathbb{R}$. Our choice of x coordinate will be

$$x_d := d - \sigma,$$

which has fractional part frac$(-\sigma)$. The motivation for choosing this value for x_d is from the next lemma, which holds for any strongly regular graph.

Lemma 3.14. *Let Γ be in SRG(v, k, λ, μ) with restricted eigenvalues $\rho > \sigma$. Then the corresponding regular adjacency polynomial is negative at $(x_d, Haem_{\geq}(\Gamma, d)+1)$. More precisely, we have*

$$R_\Gamma(x_d, Haem_{\geq}(\Gamma, d) + 1) = \sigma(2\rho + 1 - \sigma) + d(\sigma - \rho).$$

Proof. We prove this by using Maple, and the calculation can be found in Appendix 3.A. □

First we observe some useful facts about parameters of type I graphs and $\mathrm{Haem}_{\geq}(\Gamma, d)$ in the case of type I graphs.

Lemma 3.15. *Let Γ be in SRG(v, k, λ, μ) and of type I, with restricted eigenvalues $\rho > \sigma$. Then $\mu = \sigma(\sigma + 1)$.*

Proposition 3.16. *Let Γ be in SRG(v, k, λ, μ) and of type I, with restricted eigenvalues $\rho > \sigma$. Then*

$$Haem_{\geq}(\Gamma, d) = 2d - 2\sigma + \frac{d}{\sigma} - 1.$$

Proof. We have

$$k - \sigma = \frac{v-1}{2} + \frac{\sqrt{v}+1}{2} = \sqrt{v}\frac{(\sqrt{v}+1)}{2} = -\sigma\sqrt{v}$$

$$v - k + \sigma = v + \sigma\sqrt{v} = \sqrt{v}\left(\sqrt{v} - \frac{\sqrt{v}+1}{2}\right) = \rho\sqrt{v}$$

. By using these, we can deduce the following.

$$\begin{aligned}
\mathrm{Haem}_{\geq}(\Gamma, d) - d + \sigma &= (d - \sigma)\left(\frac{v}{k - \sigma} - 1\right) \\
&= (d - \sigma)\left(\frac{v - k + \sigma}{k - \sigma}\right) \\
&= (d - \sigma)\left(-\frac{\rho}{\sigma}\right).
\end{aligned}$$

Finally, using $\rho = -\sigma - 1$, we see that

$$\begin{aligned}
\text{Haem}_{\geq}(\Gamma, d) &= (d-\sigma)\left(1 - \frac{\rho}{\sigma}\right) \\
&= (d-\sigma)\left(2 + \frac{1}{\sigma}\right).
\end{aligned}$$

We also verify this using Maple in Appendix 3.A. $\square$

One can see that in the type I case, for specific values of d it may be easier to analyse R around $(x_d, \text{Haem}_{\geq}(\Gamma, d) + 1)$.

Example 3.17. For $d = \mu = k/2$, we have $u_d = 2\mu - \sigma$ and the fractional parts are the same. Thus we can compute $R_\Gamma(x_d - t, \text{Haem}_{\geq}(\Gamma, d) - t)$ and hope that we can prove this is negative for $t \in [0, 1)$. In Appendix 3.A, we use Maple to show that

$$\begin{aligned}
R_\Gamma(x_d - t, \text{Haem}_{\geq}(\Gamma, d) + 1 - t) &= (1-t)(\sigma^2 - t)(2\sigma + t) \\
R_\Gamma(x_d - t, \text{Haem}_{\geq}(\Gamma, d) - t) &= -t(\sigma^2 - t + 1)(2\sigma + t + 1)
\end{aligned}$$

The first polynomial is negative for all $t \in [0, 1)$, which reaffirms the result of Theorem 3.8. However, the second polynomial is non-negative for all $t \in [0, 1)$, so does not help us improve on the known upper bounds. $\triangle$

In general, the fractional parts are not related by any linear equation. Analysis of the fractional parts is harder when $\frac{d}{\sigma}$ is relatively large. But when $\frac{d}{\sigma}$ is small, we have the following convenient relationship.

$$\text{frac}(-2\sigma + d/\sigma) = \frac{d}{\sigma} + \text{frac}(-2\sigma) \text{ if and only if } \text{frac}(-2\sigma) > -\frac{d}{\sigma}.$$

Now we will show that for these larger fractional parts of -2σ, we can find a range of values in which the raub is at most $\text{Haem}_{\geq}(\Gamma, d) - 1$. We use the following factorisation of the rap around the point $(x_d, \text{Haem}_{\geq}(\Gamma, d))$.

Proposition 3.18. *Let Γ be in $SRG(v, k, \lambda, \mu)$ and of type I with restricted eigenvalues $\rho > \sigma$. Then*

$$R_\Gamma(x_d - t, \mathit{Haem}_{\geq}(\Gamma, d) + 1 - 2t - d/\sigma) = -(d - \sigma - t)(2t^2 - (1 - 4\sigma)t + d - 3\sigma - 1) \quad (3.4)$$

Proof. We prove this by using Maple, and the calculation can be found in Appendix 3.A. □

Consider the case in which $\text{frac}(-2\sigma + d/\sigma) = d/\sigma + \text{frac}(-2\sigma)$ and $\text{frac}(-\sigma) > 1/2$. Then $\text{frac}(-2\sigma) = 2\text{frac}(-\sigma) - 1$, and evaluating the above factorisation of the rap with $t = \text{frac}(-\sigma)$, we see that we are calculating $R_\Gamma(\lfloor x_d \rfloor, \lfloor \text{Haem}_\geq(\Gamma, d) \rfloor)$. By looking at the quadratic in (3.4), we get the following result.

Proposition 3.19. *Let Γ be in $SRG(v, k, \lambda, \mu)$ and of type I with restricted eigenvalues $\rho > \sigma$, and d be a non-negative integer where $d \leq k$. If*

$$\frac{1}{2} + \frac{d}{\sqrt{v}+1} < frac(-\sigma) < \frac{3}{4} + \frac{\sqrt{v} - \sqrt{v - 2d + 5/4}}{2},$$

then $R_\Gamma(\lfloor x_d \rfloor, \lfloor Haem_\geq(\Gamma, d) \rfloor) < 0$.

Proof. By the lower bound, we have $\text{frac}(-\sigma) > 1/2$ and $\text{frac}(-2\sigma) > -d/s$.

Let $t = \text{frac}(-\sigma)$. We calculate the discriminant Δ of the quadratic part of Equation (3.4). This gives us $\Delta = 16\sigma^2 + 16\sigma - 8d + 9$. Using type I parameter conditions we reduce this to $\Delta = 4v - 8d + 5$. As $\sigma \leq -1$ and $t \in (1/2, 1)$ we have $x_d - t > 0$. So if t is less than the smallest zero of the quadratic (3.4), we have proven that $R_\Gamma(\lfloor x_d \rfloor, \lfloor \text{Haem}_\geq(\Gamma, d) \rfloor) < 0$. But the smallest zero is precisely the assumed upper bound, seen by direct calculation. □

Uniform distribution and primes

In Greaves and Soicher [38], the following conjecture is stated as a proven result.

Conjecture 1. *Consider the set $\mathcal{P}_1 = \{p \text{ } prime; p \equiv 1 \pmod 4\}$. Then $\{\sqrt{p}/2; p \in \mathcal{P}_1\}$ is uniformly distributed modulo 1.*

After contact with the authors, we have not been able to find a reference or proof for this result. However, Conjecture 1 seems to be true and a proof of which could come from known techniques in analytic number theory. Currently we have asked an analytic number theorist for help on resolving this problem.

If Conjecture 1 is true, we can give an argument to prove for each non-negative integer d, there exists infinitely many strongly regular graphs for which $\text{Rab}_\geq(\Gamma, d) < \text{Haem}_\geq(\Gamma, d)$.

By the definition of uniform distribution, there are infinitely many primes $p \in \mathcal{P}_1$ such that $\sqrt{p}/2$ lies within any open interval contained in $(0, 1)$. Using this fact, we can prove there are infinitely many Paley graphs Γ such that $\mathrm{Rab}_{\geq}(\Gamma, d) < \mathrm{Haem}_{\geq}(\Gamma, d)$, for any fixed non-negative integer d.

Proposition 3.20. *Let d be a non-negative real number. If Conjecture 1 is true, then there are infinitely many primes p, with $p \equiv 1 \pmod 4, p \geq 2d + 1$ and*

$$\frac{d}{\sqrt{p}+1} < frac\left(\frac{\sqrt{p}}{2}\right) < min\left(\frac{1}{4} + \frac{\sqrt{p} - \sqrt{p - 2d + 5/4}}{2}, \frac{1}{2}\right)$$

Proof. For v large enough, we can take an interval (a, b) with

$$\frac{d}{\sqrt{v}+1} < a < b < \min\left(\frac{1}{4} + \frac{\sqrt{v} - \sqrt{v - 2d + 5/4}}{2}, \frac{1}{4}\right).$$

By the the assumption that Conjecture 1 holds, we find infinitely many primes p, with $p \equiv 1 \pmod 4$ and $\mathrm{frac}(\sqrt{p}/2) \in (a, b)$. Call this set of primes $\mathcal{Q}$. Of course, infinitely many of them must be in $\mathcal{S} = \mathcal{Q} \cap \{p \in \mathcal{P}; p \geq \max(2d + 1, v)\}$. Note that $d/(\sqrt{v} + 1)$ is decreasing in v, and

$$\frac{1}{2}(\sqrt{v} - \sqrt{v - 2d + 5/4})$$

is either increasing and negative, or decreasing and always positive. Either way, we must have

$$\frac{d}{\sqrt{p}+1} \leq \frac{d}{\sqrt{v}+1} < a$$

and

$$b < \min\left(\frac{1}{4} + \frac{\sqrt{v} - \sqrt{v - 2d + 5/4}}{2}, \frac{1}{4}\right) \leq \min\left(\frac{1}{4} + \frac{\sqrt{p} - \sqrt{p - 2d + 5/4}}{2}, \frac{1}{2}\right)$$

Thus the set $\mathcal{S}$ is a set of primes of the desired form. □

Theorem 3.21. *Let d be a non-negative integer. If Conjecture 1 is true, then there exists infinitely many parameter tuples (v, k, λ, μ) such that $SRG(v, k, \lambda, \mu) \neq \emptyset$, and* *for all* $\Gamma \in SRG(v,k,\lambda,\mu)$*, we have*

$$Rab_{\geq}(\Gamma,d) < Haem_{\geq}(\Gamma,d).$$

Proof. From Proposition 3.20, there are infinitely many primes p, with $p \equiv 1 \pmod 4$, $p \geq 2d+1$ and

$$\frac{1}{2} + \frac{d}{\sqrt{p}+1} < \operatorname{frac}\left(\frac{\sqrt{p}}{2}\right) + \frac{1}{2} < \min\left(\frac{3}{4} + \frac{\sqrt{p} - \sqrt{p-2d+5/4}}{2}, 1\right)$$

Call this set of primes $\mathcal{S}$, and consider a tuple $(p,(p-1)/2,(p-5)/4,(p-1)/4)$ where $p \in \mathcal{S}$. We have $\mathrm{SRG}(p,(p-1)/2,(p-5)/4,(p-1)/4) \neq \emptyset$ as the Paley graph on p vertices is in this set.

As $\operatorname{frac}(\sqrt{p}/2) < 1/2$, we have

$$\operatorname{frac}\left(\frac{\sqrt{p}}{2}\right) + \frac{1}{2} = \operatorname{frac}\left(\frac{\sqrt{p}}{2} + \frac{1}{2}\right).$$

Any graph Γ in $\mathrm{SRG}(p,(p-1)/2,(p-5)/4,(p-1)/4)$ has smallest eigenvalue $\sigma = -(\sqrt{p}+1)/2$ (see Proposition 3.13). By Proposition 3.19 we see that the $\mathrm{Rab}_{\geq}(\Gamma,d) < \mathrm{Haem}_{\geq}(\Gamma,d)$. □

3.5 The Clique Adjacency Bound

An early application of the block intersection polynomial can be found in Soicher [53]. In this paper, Soicher derives a bound for the order of cliques in edge-regular graphs. Here, we present the main tools found in Soicher [53] and Greaves and Soicher [38], and then investigate these tools computationally.

Let $\Gamma \in \mathrm{ERG}(v,k,\lambda)$ and let S be a clique in Γ. Soicher defines the *clique adjacency polynomial*, $C_\Gamma(x,y)$ as

$$C_\Gamma(x,y) := (v-y)x(x+1) - 2xy(k-y+1) + y(y-1)(\lambda-y+2).$$

This is a block intersection polynomial (2.3), where $\lambda_j = \lambda_j(\Gamma,S)$ for $i = 0,1,2$.

We can now define the *clique adjacency bound* for Γ, denoted by $\mathrm{CAB}(\Gamma)$, to be the least integer $y \geq 2$ such that there exists an integer m where $C_\Gamma(m, y+1) < 0$. The non-negativity property of the block intersection polynomials found in Proposition 2.2 and the fact that a subset of a clique is again a clique means that this is a bound on the order of a clique in Γ.

For strongly regular graphs, the clique adjacency polynomial is very closely related to the regular adjacency polynomial. Let $\Gamma \in \mathrm{SRG}(v, k, \lambda, \mu)$. Note that $\overline{\Gamma}$ is also a strongly regular graph, with the parameters of $\overline{\Gamma}$ determined by the parameters of Γ (see Brouwer, Cohen and Neumaier [14]). The following result shows a relation between the regular adjacency polynomial of $\overline{\Gamma}$ and the clique adjacency polynomial of Γ.

Proposition 3.22. $R_{\overline{\Gamma}}(x, y, 0) = C_\Gamma(y - x - 1, y)$.

Proof. This can be directly verified using the identities in Proposition 3.1. $\square$

Therefore, there exists an integer m such that $R_{\overline{\Gamma}}(m, y, 0) < 0$ if and only if there exists an integer m^* such that $C_\Gamma(m^*, y) < 0$. By observing the fact that a clique in Γ is an independent set in $\overline{\Gamma}$, we see that we can derive the clique adjacency bound from $R_{\overline{\Gamma}}$. Thus for strongly regular graphs, Proposition 3.19 is a generalisation of Greaves and Soicher [38, Theorem 1]. Note that the regular adjacency polynomial is a quadratic polynomial in both x and y, whereas the clique adjacency polynomial is a cubic polynomial in y. This suggests the regular adjacency polynomial may be easier to analyse and use in computations, when we are studying strongly regular graphs.

In [38], Greaves and Soicher compare the clique adjacency bound to the well-known Delsarte bound [27]. They prove the clique adjacency bound is at least as good as the Delsarte bound for any strongly regular graph, and is often better.

Greaves and Soicher [38] also remark on how tight the clique adjacency bound is for small strongly regular graphs. In particular, they ask

(Q) Do there exist strongly regular graphs with parameters (v, k, λ, μ), with $k < v/2$, such that every strongly regular graph having those parameters has clique number less than the clique adjacency bound.

We will show that any parameter tuple (v, k, λ, μ) with the properties in question (Q) must have $v > 40$. In other words, we claim the following.

Proposition 3.23. *There does not exist strongly regular graphs with parameters (v, k, λ, μ), with $v < 41, k < v/2$, such that every strongly regular graph having those parameters has clique number less than the clique adjacency bound.*

Proof. To do this, we will use GAP and the AGT package (Algebraic Graph Theory). A version of the manual of the AGT package can be found in Appendix A.

First, we load GAP and the AGT package. We find the primitive strongly regular graph parameters (v, k, λ, μ) such that $v < 41, 2k < v$ by using the `SmallFeasibleSRGParameterTuples` function (for the definition of primitive parameters, see A.5.4). Within these, we also find the parameter tuples for which all non-isomorphic graphs are not necessarily stored in the package using `IsAllSRGsStored`, and find how many graphs with these parameters are stored by using `NrSRGs`.

```
gap>  AGT_small:=SmallFeasibleSRGParameterTuples(40);;
gap>  AGT_small:=Filtered(AGT_small,x->2*x[2]<x[1]);;
gap>  notall:=Filtered(AGT_small,x->(not IsAllSRGsStored(x)));
[ [ 37, 18, 8, 9 ] ]
gap> NrSRGs([37,18,8,9]);
6760
```

Therefore, with the exception of the parameters $(37, 18, 8, 9)$, this package stores all primitive strongly regular graphs on at most 40 vertices. Currently, the AGT package stores 6760 non-isomorphic graphs belonging to $\mathrm{SRG}(37, 18, 8, 9)$, but the graphs with these parameters have not yet been enumerated.

We now loop through the parameters in `AGT_small`. For each parameter tuple `parms`, we first calculate the corresponding clique adjacency bound using the function `CliqueAdjacencyBound`. Then we iterate over the graphs with parameters `parms` that are stored by the AGT package, using function `SRGIterator`. If we find a graph containing a clique of order equal to the clique adjacency bound, we continue to the next set of parameters. Otherwise, we print the parameters.

```
gap> for parms in AGT_small do
>   isTight:=false;
>   cab:=CliqueAdjacencyBound(parms{[1,2,3]});
>   srgs:=SRGIterator(parms);
>   for gamma in srgs do
```

```
>     if CliqueNumber(gamma)=cab then
>       isTight:=true;
>       break;
>     fi;
>   od;
>   if isTight=false then
>     Print("CAB is not tight for parameters:",parms,"\n");
>   fi;
> od;
gap>
```

This loop took around 4 minutes on a personal laptop, and resulted in no parameter tuples being printed. This proves Proposition 3.23. □

3.A Verification with Maple

In this section, polynomial identities used in this chapter are verified through the use of Maple [8].

We start Maple and define R as the regular adjacency polynomial.

```
> R:=x*(x+1)*(v-y)-2*x*y*k+(2*x+rho+sig+1)*y*d+y*(y-1)*mu-y*d^2;
```

To work with the parameters of strongly regular graphs, we will use the Maple package `Groebner`. With the parameters $(v, k, lambda, mu)$ and restricted eigenvalues rho, sig, we will first define the polynomial ring

$$P = \mathbb{Q}[t, d, v, k, lambda, mu, rho, sig],$$

where t is considered as our indeterminate. Then the `Groebner` package will be used to calculate Gröbner bases and work in certain factor rings. For more information on Gröbner bases, see Adams and Loustaunau [1].

We derive relators from Proposition 3.1 which evaluate to 0 on the parameters $(v, k, lambda, mu)$ and restricted eigenvalues rho, sig of a strongly regular graph.

```
> srg_rel:={mu*(v-k-1)-k*(k-l-1),lambda-mu-rho-sig,mu-k-rho*sig};
```

For type I graphs, we add relators derived from the definition of their parameters.

```
> t1_rel:=srg_rel union {2*k-v+1,4*l-v+5,4*mu-v+1};
```

Now we define a monomial order in the polynomial ring.

```
> ord:=tdeg(t,d,v,k,lambda,mu,rho,sig);
```

Next we find the Gröbner bases of the ideals generated by the above relators.

```
> G:=Groebner[Basis](srg_rel,ord);
> H:=Groebner[Basis](t1_rel,ord);
```

Let I be the ideal generated by the strongly regular graph relators `srg_rels`, and J be the ideal generated by the type 1 relators `t1_rels`. The Gröbner package can then be used to do calculations in P modulo the ideals I and J.

Now we verify the various identities and polynomials found in this chapter. Some by-hand proofs have been provided in the text, but we provide a proof using Maple to verify their correctness.

First we check Proposition 3.2.

```
> Groebner[NormalForm](v*mu-(k-sig)*(k-rho),G,ord);
0
```

The next three identities are used in the results of Section 3.4. We verify the first equation from Proposition 3.8.

```
> Groebner[NormalForm](-(v-y)*factor(eval(
> R,[x=((k-d+1)*y-v)/(v-y)]))/y
> -(mu*y^2-((d-rho)*(k-sig)+(k-rho)*(d-sig))*y+v*(d-sig)*(d-rho)),G,ord);
0
```

Next we verify the factorisation used later in Proposition 3.8.

```
> Groebner[NormalForm](-(v-y)*mu*factor(eval(
> R,[x=((k-d+1)*y-v)/(v-y)]))/y
>  -((mu*y-(d-rho)*(k-sig))*(mu*y-(d-sig)*(k-rho))),G,ord);
0
```

The following identity is used in Proposition 3.10.

```
> Groebner[NormalForm](eval(R,[y=v,d=k]),G,ord);
0
```

We now move to Section 3.4.2. Here is verification of Lemma 3.14.

```
> Groebner[NormalForm](expand(
> mu*eval(R,[x=d-sig,y=(k-rho)*(d-sig)/mu+1]))
>  -mu*(sig*(2*rho+1-sig)+d*(sig-rho)),G,ord);
0
```

Now we check the correctness of Proposition 3.16.

```
> Groebner[NormalForm](expand(
> s*(v*(d-sig)-(k-sig)*(d-sig)*(2+1/sig))),H,ord);
0
```

The next two identities are found in Example 3.17.

```
> Groebner[NormalForm](eval(R,[x=mu-sig-t,y=2*mu-sig+1-t,d=mu])
> -(1-t)*(sig^2-t)*(2*sig+t),H,ord);
0
> Groebner[NormalForm](eval(R,[x=mu-sig-t,y=2*mu-sig-t,d=mu])
> -(-t)*(sig^2-t+1)*(2*sig+t+1),H,ord);
0
```

Finally we check Proposition 3.18.

```
> Groebner[NormalForm](eval(R,[x=d-sig-t,y=2*d-2*sig-2*t])
>  +(d-sig-t)*(2*t^2-(1-4*sig)*t+d-3*sig-1),H,ord);
0
```

Chapter 4

Regular cliques in edge-regular graphs

In the early 1980s, A. Neumaier [49] studied regular cliques in edge-regular graphs, and a certain class of designs whose point graphs are strongly regular and contain regular cliques. In his investigations, all of the edge-regular graphs with regular cliques which he encountered were strongly regular. Further, he proved that any vertex-transitive and edge-transitive graph containing a regular clique is strongly regular. He then posed the problem:

(P) [49] Is every edge-regular graph with a regular clique strongly regular?

We thus define a *Neumaier graph* to be a non-complete edge-regular graph containing a regular clique and define a *strictly Neumaier graph* to be a non-strongly regular Neumaier graph.

Informed about the problem by L. Soicher in 2015, G. Greaves and J. Koolen then gave an answer by constructing an infinite family of strictly Neumaier graphs [37]. A. Gavrilyuk and S. Goryainov then searched for examples in a collection of known Cayley-Deza graphs [35], leading to the discovery of four more strictly Neumaier graphs, each having order 24. Recently, Greaves and Koolen [36] presented another infinite family of strictly Neumaier graphs, which contains one of the four strictly Neumaier graphs found by Gavrilyuk and Goryainov.

In their paper [37], Greaves and Koolen pose two further questions about strictly Neumaier graphs which naturally arose from their work:

(A) [37, Question A] What is the minimum number of vertices for which there exists a non-strongly-regular, edge-regular graph having a regular clique?

(B) [37, Question B] Does there exist a non-strongly-regular, edge-regular graph having a regular clique with nexus greater than 1?

Indeed, before our work, all known strictly Neumaier graphs had at least 24 vertices, and contained m-regular cliques only for the value $m = 1$.

Further to a discussion with Koolen, Goryainov and his student D. Panasenko found the smallest strictly Neumaier graph, using methods similar to some of their work on Deza graphs. At roughly the same time, the author found the smallest strictly Neumaier graph in a collection of vertex-transitive edge-regular graphs which had been provided by G. Royle [42]. Subsequent communications led to the collaboration found in the paper of Goryainov, Panasenko and the author [31].

In this chapter we answer both of the above questions. We first give some general results on Neumaier graphs and their feasible parameter tuples. In particular, we concentrate on conditions involving parameter tuples that force a Neumaier graph to be strongly regular. We also give a classification of Neumaier graphs with parameters achieving equality in a certain inequality. Then we apply these results to determine the smallest strictly Neumaier graph, which turns out to be vertex-transitive and has order 16, valency 9 and contains a 2-regular 4-clique.

4.1 Neumaier graphs

A *Neumaier graph* is a non-complete edge-regular graph which contains a regular clique. We denote by $\mathrm{NG}(v, k, \lambda; m, s)$ the set of Neumaier graphs which are edge-regular with parameters (v, k, λ), and contain an m-regular s-clique, where $s \geq 2$. A *strictly Neumaier graph* is a Neumaier graph which is not strongly regular (the definition of a strictly Neumaier graph is analoguous to the definition of a strictly Deza graph, see Erickson et al. [29]).

Let us give several examples of strongly regular graphs which are also Neumaier graphs.

Example 4.1. Let $K_{r\times t}$ be the complete multipartite graph which has r parts of size

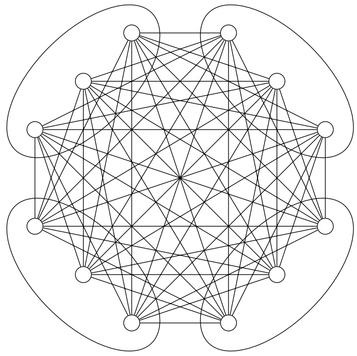

Figure 4.1: The complete multipartite graph $K_{4\times 3}$.

t. Let S be a set consisting of exactly one vertex from each part of Γ. Then S is a $(r-1)$-regular r-clique. △

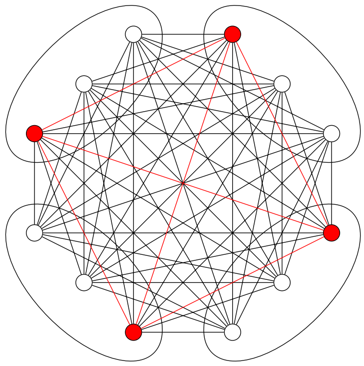

Figure 4.2: A 3-regular 4-clique in $K_{4\times 3}$.

Example 4.2. In Example 3.5, we introduced the square lattice graph $L_2(n)$, which is strongly regular with parameters $(n^2, 2(n-1), n-2, 2)$. Let S be a set consisting of all vertices of $L_2(n)$ which have the same fixed value at the same fixed coordinate. Then S is a 1-regular n-clique. △

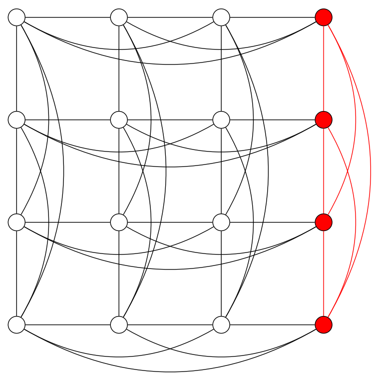

Figure 4.3: A 1-regular 4-clique in $L_2(n)$.

Example 4.3. For $n \geq 3$, the *triangular graph* $T(n)$ has vertex set consisting of the subsets of $\{1, 2, ..., n\}$ of size 2, and two distinct vertices A, B are joined by an edge precisely when $|A \cap B| = 1$ (alternatively this graph can be described as the line graph of the complete graph K_n). This graph is strongly regular with parameters $(\binom{n}{2}, 2(n-2), n-2, 4)$. Let S be a set consisting of all vertices of $T(n)$ which contain a fixed element from $\{1, 2, ..., n\}$. Then S is a 2-regular $(n-1)$-clique. △

The tuple (v, k, λ) is said to be *extremal* if $\mathrm{ERG}(v, k, \lambda)$ is non-empty and contains only strongly regular graphs. Similarly, the tuple $(v, k, \lambda; m, s)$ is said to be *extremal* if $\mathrm{NG}(v, k, \lambda; m, s)$ is non-empty and contains only strongly regular graphs.

To answer Question A, we collect a series of conditions on the parameters $(v, k, \lambda; m, s)$ that force at least one of the following to occur;

(i) $\mathrm{ERG}(v, k, \lambda)$ is empty.

(ii) (v, k, λ) is extremal.

(iii) $\mathrm{NG}(v, k, \lambda; m, s)$ is empty.

(iv) $(v, k, \lambda; m, s)$ is extremal.

We will present both new and known results on extremal parameter tuples (v, k, λ) and $(v, k, \lambda; m, s)$. Conditions for a parameter tuple (v, k, λ) to be extremal is an interesting area of study, and many instances of this type of extremality condition can be found in the literature (for example, in Coolsaet et al. [23] or Brouwer, Cohen and Neumaier [14, Section 1.4]).

4.2 Parameters of Neumaier graphs

In this section, we study some necessary conditions on the parameters of a Neumaier graph. The next lemma gives basic properties of the parameters of an edge-regular graph.

Lemma 4.4. *Let* Γ *be in* $ERG(v, k, \lambda)$*. Then:*

(i) $v > k > \lambda \geq 0$*;*

(ii) $v \geq 2k - \lambda$*;*

(iii) 2 *divides* vk*;*

(iv) 2 *divides* $k\lambda$*;*

(v) 6 *divides* $vk\lambda$*.*

Proof. These are standard calculations, and can be found in Brouwer, Cohen and Neumaier [14, Chapter 1]. □

Further, we list several more tools which we use in the investigation of Neumaier graphs. The next result gives arithmetic conditions on the parameters of a Neumaier graph. By analysing these relations further, we reconstruct s and m as functions of v, k, λ. The property of these expressions to be integral numbers can then be seen as necessary conditions for an edge-regular graph to contain a regular clique.

Lemma 4.5. *Let* Γ *be a graph in* $NG(v, k, \lambda; m, s)$*. Then:*

(i) $(v - s)m = (k - s + 1)s$*;*

(ii) $(k - s + 1)(m - 1) = (\lambda - s + 2)(s - 1)$*;*

(iii) s *is the largest root of the polynomial*

$$(v - 2k + \lambda)y^2 + (k^2 + 3k - \lambda - v(\lambda + 2))y + v(\lambda + 1 - k);$$

(iv) m *is the largest root of the polynomial*

$$(v - s)x^2 - (v - s)x - s(s - 1)(\lambda - s + 2).$$

Proof. Let S be an m-regular s-clique in Γ. Soicher derives the clique adjacency polynomial found in Section 3.5 by calculating $\lambda_i(\Gamma, S)$ (these calculations can found in Soicher [53, Theorem 1.1]), where

$$\begin{aligned}\lambda_0(\Gamma, S) &= v - s,\\ \lambda_1(\Gamma, S) &= k - s + 1,\\ \lambda_2(\Gamma, S) &= \lambda - s + 2.\end{aligned}$$

Then (i) follows from Equation (2.13) and (ii) follows from Equation (2.14).

(iii) Multiply the expression in (ii) by $(v-s)$ and use (i) to substitute for $(v-s)m$. We see that s is a root of the polynomial. Note that $v \geq 2k-\lambda$ and $v(\lambda+1-k) \leq 0$ by Lemma 4.4. This means there is at most one positive root to the polynomial.

(iv) Multiply the expression in (i) by $(m-1)$ and use (ii) to substitute for $(k-s+1)(m-1)$. We see that m is a root of the polynomial. Note that $\lambda-s+2 \geq 0$ as an edge in an s-clique is in at least $s-2$ triangles of the graph Γ. This means there is at most one positive root of the polynomial. □

Now we present a collection of results giving properties of all regular cliques in a Neumaier graph.

Lemma 4.6. *Let Γ be a graph in $NG(v, k, \lambda; m, s)$. Then:*

(i) the maximum size of a clique in Γ is s;

(ii) all regular cliques in Γ are m-regular cliques;

(iii) the regular cliques in Γ are precisely the cliques of size s.

Proof. This is a simple calculation, the proof of which can be found in Neumaier [49, Theorem 1.1]. □

4.3 Forcing strong regularity

We will now give a collection of conditions on parameter tuples to show they are extremal. We first consider the tuples associated with edge-regular graphs, and then consider the tuples associated with Neumaier graphs.

4.3.1 The triple of parameters (v, k, λ)

When the triple (v, k, λ) is extremal, there are no edge-regular graphs in ERG(v, k, λ) which are not strongly regular. Thus there is no strictly Neumaier graph with these edge-regular parameters. This fact will be heavily used when analysing the smallest strictly Neumaier graph.

The following lemma gives a list of sufficient conditions for (v, k, λ) to be extremal.

Lemma 4.7. *Suppose ERG(v, k, λ) is non-empty for some v, k, λ. Then the triple (v, k, λ) is extremal if at least one of the following holds:*

(i) $v = 2k - \lambda$.

(ii) $v = 2k - \lambda + 1$.

(iii) There is a strongly regular graph with parameters $(v, v - k - 1, 0, v - 2k + \lambda)$.

Proof. (i) Using the block intersection polynomial derived from the above λ_i, Soicher proves that an edge-regular graphs has parameters satisfying this equality if and only if it is isomorphic to $K_{s \times t}$, for some s, t [54, Theorem 4.1] (Soicher also characterises these graphs as the edge-regular graphs which have certain quasiregular cliques).

(ii) Take an edge-regular graph with parameters (v, k, λ), with $v - 2k + \lambda = 1$. By Brouwer, Cohen and Neumaier [14, Section 1.1], we see that $\overline{\Gamma}$ is co-edge-regular with parameters $(v, v - k - 1, 1)$. Then by [14, Lemma 1.1.3], $\overline{\Gamma}$ is strongly regular. Thus Γ is strongly regular.

(iii) Let Δ be a strongly regular graph with parameters $(v, v - k - 1, 0, v - 2k + \lambda)$. By [14, Theorem 1.3.1], $\overline{\Delta}$ is strongly regular with parameters (v, k, λ, μ). In particular, we note that by Proposition 3.1 (i), we have $k(k - \lambda - 1) = \mu(v - k - 1)$.

Now let $u \in V(\Gamma)$. First we partition $V(\Gamma)$ into $V_1 = \{u\}, V_2 = \Gamma(u)$ and $V_3 = V(\Gamma) \setminus (\Gamma(u) \cup \{u\})$. Since each vertex in V_2 has $k - \lambda - 1$ neighbours in V_3, there are $k(k - \lambda - 1)$ edges between V_2 and V_3.

Define b as the average number of neighbours a vertex in V_3 has in V_2. Then the number of edges between V_3 and V_2 is $b(v - k - 1)$. Therefore, we have $k(k - \lambda - 1)$ is equal to $b(v - k - 1)$ and $\mu(v - k - 1)$, so $b = \mu$.

Let $w \in V_3$. The number of neighbours of w in V_2 is at least $k-(v-k-2)=\mu$, as $|V_3| = v-k-1$. As $b=\mu$ is the average of numbers at least as big as b, they must all equal b. This means the number of common neighbours of u and w in Γ is exactly μ, and Γ is strongly regular with parameters (v,k,λ,μ). We can also see that in this case, $\mu = 2k+2-v$. $\square$

4.3.2 The quintuple of parameters $(v,k,\lambda;m,s)$

Next we will give a necessary conditions on the parameters of a graph in $\mathrm{NG}(v,k,\lambda;m,s)$ to be extremal.

We start by giving a lower bound on the size of a regular clique in a strictly Neumaier graph. We can understand such a result as saying the following: Take a parameter tuple $(v,k,\lambda;m,s)$ where s is less than the bound. Then the parameter tuple $(v,k,\lambda;m,s)$ is extremal.

Lemma 4.8. *Let Γ be a strictly Neumaier graph from NG$(v,k,\lambda;m,s)$. Then $s \geq 4$, and consequently, $\lambda \geq 2$.*

Proof. The lower bound on the clique size can be shown by simple counting arguments, and a proof is given in Greaves and Koolen [37, Proposition 5.1]. The bound on λ follows immediately. $\square$

Next we give a necessary condition, in the form of an inequality that is linear in the parameters k,λ,m,s. When equality is achieved, we show that the parameter tuple $(v,k,\lambda;m,s)$ is extremal.

To prove this result, we first give a useful lemma involving Neumaier graphs where the neighbourhood of any vertex has a certain structure.

Lemma 4.9. *Let Γ be a graph from NG$(v,k,\lambda;m,s)$. Further suppose that every vertex in Γ has neighbourhood consisting of l vertex disjoint cliques of size $s-1$. Then Γ is strongly regular, with parameters $v = s+(l-1)(s-1)s/m$, $k = l(s-1)$, $\lambda = (m-1)(l-1)+s-2$ and $\mu = lm$.*

Proof. Take any vertex $u \in V(\Gamma)$ and $w \notin \Gamma(u)$. The neighbourhood of u consists of disjoint $(s-1)$-cliques. Together with u each of these cliques define an s-clique. These cliques are necessarily m-regular by Lemma 4.6. Thus w is adjacent to m

vertices in each of these cliques, and has exactly lm neighbours in common with u. This proves Γ is strongly regular with $\mu = lm$.

The formulae for k and λ can be derived by simple counting arguments. Then for v, we use Lemma 4.5.

□

Now we give the inequality of the parameters $(v, k, \lambda; m, s)$ of a Neumaier graph.

Theorem 4.10. *Let Γ be a graph from $NG(v, k, \lambda; m, s)$. Then*

$$k - \lambda - s + m - 1 \geq 0 \qquad (*)$$

Equality holds if and only if every vertex in Γ has a neighbourhood consisting of two vertex disjoint $(s-1)$-cliques. In this case, Γ is strongly regular with $v = s + (s(s-1)/m)$, $k = 2(s-1)$, $\lambda = s + m - 3$ and $\mu = 2m$.

Proof. Let S be an m-regular s-clique in Γ and $u \in S$. Consider a vertex $w \in V(\Gamma) \setminus S$, with $uw \in E(\Gamma)$. We know that u has $k - s$ other neighbours in $V(\Gamma) \setminus S$, and w has $m - 1$ neighbours in $S \setminus \{u\}$. Thus u and w have exactly $m - 1$ common neighbours in S, and at most $k - s$ common neighbours in $V(\Gamma) \setminus S$. As u, w have exactly λ common neighbours, we must have $\lambda \leq k - s + m - 1$.

When equality holds, we see that w must be adjacent to all neighbours of u in $V(\Gamma) \setminus S$. By repeating the argument for all other edges uz, with $z \in V(\Gamma) \setminus S$, we see that u has a neighbourhood consisting of two vertex disjoint cliques.

By Lemma 4.5 (ii) and $k = \lambda + s - m + 1$, we deduce that $(\lambda - s - m + 3)(s - m) = 0$. If $s = m$, Γ is necessarily complete. Otherwise, $\lambda = s + m - 3$ and $k = 2(s-1)$. This proves that for all $u \in S$, u has a neighbourhood consisting of two vertex disjoint $(s-1)$-cliques.

Now take a vertex $u \in V(\Gamma) \setminus S$. As $m \geq 1$, u is adjacent to a vertex $w \in S$. As the neighbourhood of w consists of $(s-1)$-cliques, u is contained in S', which is one of these $(s-1)$-cliques. Then $S = S' \cup \{w\}$ is an s-clique that contains u. Thus, we have proved that every vertex is contained in an s-clique.

By Lemma 4.6, any s-clique is necessarily m-regular. So we can apply the above argument to show that any vertex $u \in V(\Gamma)$ has a neighbourhood consisting of two vertex disjoint $(s-1)$-cliques in Γ. The result then follows from Lemma 4.9. □

4.3.3 Classifying the graphs in the equality case

We now classify all Neumaier graphs for which their parameters are in the equality case of Theorem 4.10.

Theorem 4.11. *Let Γ be a graph from $NG(v, k, \lambda; m, s)$, where $k-\lambda-s+m-1 = 0$. Then Γ is one of the following strongly regular graphs:*

(i) the square lattice graph $L_2(s)$;

(ii) the triangular graph $T(s+1)$, where $s \geq 3$;

(iii) the complete s-partite graph $K_{s\times 2}$, with parts of size 2.

We prove this theorem by taking cases on the value of m. We start with the case $m = 1$. In this case we will give two proofs. The first argues that there is at most one Neumaier graph with the desired properties up to isomorphism, and as the graph $L_2(n)$ also has these properties, we are done. The second uses the fact that all strongly regular graphs with the desired properties have been classified, and then checks all of the graphs in question.

Lemma 4.12. *Let Γ be a graph from $NG(v, k, \lambda; m, s)$, where $k - \lambda - s + m - 1 = 0$ and $m = 1$. Then Γ is isomorphic to the square lattice graph $L_2(s)$.*

Proof 1. Assume Γ in non-complete. By Theorem 4.10, we see that Γ is edge-regular with parameters $(s^2, 2(s-1), s-2)$.

Let $S \subset V(\Gamma)$, $S = \{u_1, u_2, \ldots, u_s\}$, be a 1-regular s-clique in Γ. As $k = 2(s-1)$ and S is 1-regular, $V(\Gamma)\backslash S$ can be partitioned into disjoint sets of vertices $V_i = \{u_{ij} : j \in \{1, \ldots, s-1\}\}$ for $i \in \{1, ..., s\}$, where u_i is adjacent to u_{ij} for all $j \in \{1, \ldots, s-1\}$. As we are in the equality case of Theorem 4.10, the sets V_i are themselves cliques of size $s-1$ and $V_i \cup \{u_i\}$ are cliques of size s.

Take a vertex $u_{ij} \in V_i$. By Lemma 4.6, every s-clique is 1-regular. Therefore, each u_{ij} is adjacent to a unique vertex in V_k, for $k \neq i$. Without loss of generality, we can assume u_{ij} is adjacent to $\{u_{kj} : k \neq i\}$. If u_{kj} is not adjacent $u_{k'j}$ for $k' \neq k$, then $u_{ij}u_{kj}$ cannot be in more than $s-3$ triangles. Thus $\{u_{kj} : k \neq i\}$ is a clique of size $s-1$. So u_{kj} is adjacent to the vertices $V_k\backslash\{u_{kj}\} \cup \{u_{k'j} : k' \neq k\} \cup \{u_k\}$, which is a set of size $(s-2)+(s-1)+1=2(s-1)$. Thus the neighbourhood of u_{kj} is completely determined.

We continue this argument with different i, j, until we have completely determined the graph (up to isomorphism). As $L_2(s)$ is edge-regular with these parameters and contains a 1-regular s-clique, the result follows. □

Proof 2. By Theorem 4.10, Γ is strongly regular with parameters $(s^2, 2(s-1), s-2, 2)$. Any strongly regular graph with parameters $(s^2, 2(s-1), s-2, 2)$ must be isomorphic to $L_2(s)$, unless $s=4$ (this was the aim of the paper of Shrikhande [52]).

Here we can use the AGT package (see Appendix A) to investigate the strongly regular graphs in $\mathrm{SRG}(16,6,2,2)$. After loading GAP, we first check that the AGT package stores all strongly regular graphs in $\mathrm{SRG}(16,6,2,2)$.

```
gap> parms:=[16,6,2,2];;
gap> IsAllSRGsStored(parms);
true
```

Now we know that all graphs in $\mathrm{SRG}(16,6,2,2)$ are available in the AGT package, we can check if each of them is a Neumaier graph using the `IsNG` function.

```
gap> lem_12:=AllSRGs([16,6,2,2]);
gap> List(lem_12,IsNG);
[ true, false ]
```

Therefore, the graph `lem_12[1]` is the only Neumaier graph which is in $\mathrm{SRG}(16,6,2,2)$. We can then check that this graph is isomorphic to the graph $L_2(4)$.

```
gap> L24:=SquareLatticeGraph(4);;
gap> IsIsomorphicGraph(lem_12[1],L24);
true
```

In fact, it is well-known that there is only one strongly regular graph in $\mathrm{SRG}(16,6,2,2)$ that is not isomorphic to $L_2(4)$, and this graph is called the *Shrikhande graph* (this was the aim of the paper of Chang [20]). □

Next we consider the case $m=2$. In this case we will give two proofs. Like in the above case, the first argues that there is at most one Neumaier graph with the desired properties up to isomorphism, and as the graph $T(n)$ also has these properties, we are done. The second uses the fact that all strongly regular graphs with the desired properties have been classified, and then checks all of the graphs in question.

Lemma 4.13. *Let* Γ *be a graph from* $NG(v, k, \lambda; m, s)$*, where* $k - \lambda - s + m - 1 = 0$ *and* $m = 2$*. Then* Γ *is isomorphic to the triangular graph* $T(s+1)$*.*

Proof 1. Assume Γ in non-complete. By Theorem 4.10, we see that Γ is edge-regular with parameters

$$\left(\binom{s+1}{2}, 2(s-1), s-1\right).$$

Let $S \subset V(\Gamma)$, $S = \{u_1, u_2, \ldots, u_s\}$, be a 2-regular s-clique in Γ. Any edge u_iu_j in Γ is in $s-2$ triangles with 3rd vertex in S. If any 2 vertices outside of S are adjacent to the same vertices u_i, u_j in K, u_iu_j is in at least $s - 2 + 2 = s$ triangles. This contradicts $\lambda = s - 1$. Thus every vertex in $V(\Gamma)\backslash S$ has a different set of neighbours in S. We denote the vertex of $V(\Gamma)\backslash S$ with neighbours $\{u_i, u_j\}$ in S as u_{ij}.

As we are in the equality case of Theorem 4.10, u_{ij} is adjacent to $\{u_{ik} : k \neq j\} \cup \{u_{kj} : k \neq j\} \cup \{u_i, u_j\}$. This set has size $2(s-2) + 2 = 2(s-1)$, so we have completely determined all neighbours of u_{ij} in the graph, and hence all edges in Γ. As $T(s+1)$ is edge-regular with these parameters and contains a 2-regular s-clique, the result follows. □

Proof 2. By Theorem 4.10, Γ is strongly regular with parameters $(s^2, 2(s-1), s-2, 2)$. Any strongly regular graph with parameters $(\binom{s+1}{2}, 2(s-1), s-1, 4)$ must be isomorphic to $T(s+1)$, unless $s = 7$ (this is a result found through the combination of the work of Connor [22], Hoffman [41], and Shrikhande [51], or as the result of Chang [20]).

Here we can use the AGT package (see Appendix A) to investigate the strongly regular graphs in SRG$(28, 12, 6, 4)$. After loading GAP, we first check that the AGT package stores all strongly regular graphs in SRG$(28, 12, 6, 4)$.

```
gap> parms:=[28,12,6,4];;
gap> IsAllSRGsStored(parms);
true
```

Now we know that all graphs in $\mathrm{SRG}(28,12,6,4)$ are available in the AGT package, we can check if each of them is a Neumaier graph using the `IsNG` function.

```
gap> lem_t8:=AllSRGs([28,12,6,4]);
gap> List(lem_t8,IsNG);
[ false, true, false, false ]
```

Therefore, the graph `lem_t8[2]` is the only Neumaier graph which is in $\mathrm{SRG}(28,12,6,4)$. We can then check that this graph is isomorphic to the graph $T(8)$.

```
gap> T8:=TriangularGraph(8);;
gap> IsIsomorphicGraph(lem_t8[2],T8);
true
```

In fact, it is well-known that there are only three strongly regular graphs in $\mathrm{SRG}(28,12,6,4)$ that are not isomorphic to $T(8)$, and these graphs are called the *Chang graphs* (this is the aim of the paper of Chang [20]). $\square$

Now we only need to consider the case $m \geq 3$. For this case, we can show that m is particularly large with respect to s, which forces the graph to be isomorphic to $K_{s\times 2}$.

Lemma 4.14. *Let Γ be a graph from $NG(v,k,\lambda;m,s)$, where $k-\lambda-s+m-1=0$ and $m \geq 3$. Then Γ is isomorphic to $K_{s\times 2}$.*

Proof. We will first show that $m \geq 1+s/2$.

Let Γ be a graph in $\mathrm{NG}(v,k,\lambda)$. Take a subset $S \subset V(\Gamma)$, $S=\{u_1,u_2,\dots,u_s\}$, where S a m-regular s-clique in Γ. Without loss of generality, let $w \in V(\Gamma)\setminus S$, with $\{u_1,u_2,u_3\} \subseteq \Gamma(w)\cap S$. Note that by the equality case of Theorem 4.10, w is adjacent to all neighbours of u_1,u_2,u_3 in $V(\Gamma)\setminus S$.

As Γ is k-regular, we have $|\Gamma(u_i)\cap V(\Gamma)\setminus S| = k-s+1$ for all $i \in \{1,\dots,s\}$. Also we must have $|\Gamma(u_i)\cap\Gamma(u_j)\cap V(\Gamma)\setminus S| = \lambda-s+2$ for all $i,j\in\{1,\dots,s\}$. Then we have

$$\begin{aligned}|(\Gamma(u_1)\cup\Gamma(u_2)\cup\Gamma(u_3))\cap V(\Gamma)\setminus S| &= 3(k-s+1)-3(\lambda-s+2)\\ &\quad+|(\Gamma(u_1)\cap\Gamma(u_2)\cap\Gamma(u_3))\cap V(\Gamma)\setminus S|\end{aligned}$$

$$\geq 3(k - \lambda - 1),$$

and therefore we have

$$k = |\Gamma(w)| \geq m + 3(k - \lambda - 1).$$

By using $\lambda = k - s + m - 1$ and $k = 2(s-1)$ (by Theorem 4.10), we have shown that $m \geq 1 + s/2$.

Let $u, w \in V(\Gamma) \setminus S$. As S is m-regular and $m > s/2$, there must exist a $i \in S$ such that $iu, iw \in E(\Gamma)$. We also know that the neighbourhood of i in $V(\Gamma) \setminus S$ is a clique, so $uw \in E(\Gamma)$. Thus we have shown that $V(\Gamma) \setminus S$ is a clique in Γ.

By maximality of S, we must have $|V(\Gamma) \setminus S| \leq s$. Also, because $k = 2(s-1)$, we must have $|V(\Gamma) \setminus S| \geq s - 1$. As Γ is non-complete, we have $|V(\Gamma) \setminus S| = s$, and $m = s - 1$. By Theorem 4.10, $v = 2s, \lambda = 2(s-2)$. Applying Proposition 4.7, we have shown the result. □

4.4 The smallest strictly Neumaier graphs

The following tables list all tuples $(v, k, \lambda; m, s)$ of integers, such that the following hold:

1. $0 < k < v - 1,\ v \leq 24,\ 0 \leq \lambda < k,\ 2 \leq s \leq \lambda + 2$ and $m \geq 1$.
2. 2 divides both vk and $k\lambda$, and 6 divides $vk\lambda$ (see Lemma 4.4).
3. $(v-s)m = (k-s+1)s$ and $(k-s+1)(m-1) = (\lambda-s+2)(s-1)$ (see Lemma 4.5 (i) and (ii)).

These tables were obtained by a straightforward computation using the AGT package in GAP [39]. All calculations were exact and took a total of about 20 CPU milliseconds on a desktop PC.

Thus, if there is a Neumaier graph from $\mathrm{NG}(v, k, \lambda; m, s)$ such that $v \leq 24$, then the tuple v, k, λ, m, s appears in our tables. The rightmost column of our tables display a result which proves that the tuple (v, k, λ) or the tuple $(v, k, \lambda; m, s)$ is extremal, or the symbol '-' otherwise. For example, L4.7 (i) refers to Lemma 4.7 part (i), and T4.10 refers to Theorem 4.10.

v	k	λ	m	s	result
4	2	0	1	2	L4.7 (i)
6	3	0	1	2	L4.7 (i)
	4	2	2	3	L4.7 (i)
8	4	0	1	2	L4.7 (i)
	6	4	3	4	L4.7 (i)
9	4	1	1	3	L4.8
	6	3	2	3	L4.7 (i)
10	5	0	1	2	L4.7 (i)
	6	3	2	4	L4.7 (iii)
	8	6	4	5	L4.7 (i)
12	5	2	1	4	T4.10
	6	0	1	2	L4.7 (i)
		4	1	6	T4.10
	8	4	2	3	L4.7 (i)
	9	6	3	4	L4.7 (i)
	10	8	5	6	L4.7 (i)

v	k	λ	m	s	result
14	7	0	1	2	L4.7 (i)
	9	6	3	7	T4.10
	12	10	6	7	L4.7 (i)
15	6	1	1	3	L4.8
		3	1	5	T4.10
	8	4	2	5	T4.10
	10	5	2	3	L4.7 (i)
		6	3	5	Lii
	12	9	4	5	L4.7 (i)
16	6	2	1	4	T4.10
	8	0	1	2	L4.7 (i)
		6	1	8	T4.10
	9	4	2	4	-
	10	6	3	6	L4.7 (iii)
	12	8	3	4	L4.7 (i)
	14	12	7	8	L4.7 (i)

Table 4.1: Possible parameters of Neumaier graphs on $v \leq 16$ vertices

We see that Table 4.1 rules out all possible parameter tuples $(v, k, \lambda; m, s)$ for a strictly Neumaier graph when $v < 16$. Further, the table shows that any strictly Neumaier graph on 16 vertices is from $\mathrm{NG}(16, 9, 4; 2, 4)$. The adjacency matrix of a strictly Neumaier graph in $\mathrm{NG}(16, 9, 4; 2, 4)$ is given in Figure 4.4 (and as Table 5.2). So Table 4.1 and this graph give the answer to Question A.

Tables 4.1 and 4.2 together show that $(24, 8, 2)$ is the only possible parameter tuple for a strictly Neumaier graph containing a 1-regular clique when $v \leq 24$.

Finally, S. Goryainov and D. Panasenko use MAGMA [10] to show that there is only one strictly Neumaier graph in $\mathrm{NG}(16, 9, 4; 2, 4)$, up to isomorphism, and there are no strictly Neumaier graphs with parameter tuples $(21, 14, 9; 4, 7)$ and $(22, 12, 5; 2, 4)$. In their computational approach, we start with the subgraph induced by vertices of a clique with given size. Then we construct all regular graphs such that the fixed clique is regular with given nexus, and check if any of the graphs are edge-regular. We finish by checking if the resulting graphs are isomorphic.

Thus we have found that any strictly Neumaier graph on at most 24 vertices must have parameters $(16, 9, 4; 2, 4)$ or $(24, 8, 2; 1, 4)$.

v	k	λ	m	s	result
18	7	4	1	6	T4.10
	9	0	1	2	L4.7 (i)
	12	6	2	3	L4.7 (i)
	15	12	5	6	L4.7 (i)
	16	14	8	9	L4.7 (i)
20	10	0	1	2	L4.7 (i)
	15	10	3	4	L4.7 (i)
	16	12	4	5	L4.7 (i)
	18	16	9	10	L4.7 (i)
21	8	1	1	3	L4.8
		5	1	7	T4.10
	10	5	2	6	T4.10
	12	7	3	7	T4.10
		8	3	9	T4.10
	14	7	2	3	L4.7 (i)
		9	4	7	-
	15	10	4	6	L4.7 (ii)
	16	12	6	9	L4.7 (ii)
	18	15	6	7	L4.7 (i)
22	11	0	1	2	L4.7 (i)
	12	5	2	4	-
	14	9	4	8	T4.10
	16	12	6	11	T4.10
	20	18	10	11	L4.7 (i)
24	8	2	1	4	-
		4	1	6	T4.10
	9	6	1	8	T4.10
	12	0	1	2	L4.7 (i)
		10	1	12	T4.10
	16	8	2	3	L4.7 (i)
	18	12	3	4	L4.7 (i)
		15	6	16	T4.10
	20	16	5	6	L4.7 (i)
		17	10	16	T4.10
	21	18	7	8	L4.7 (i)
	22	20	11	12	L4.7 (i)

Table 4.2: Possible parameters of Neumaier graphs on $16 < v \leq 24$ vertices

4.4.1 Vertex-transitive strictly Neumaier graphs

In [31], Goryainov, Panasenko and the author discovered the smallest strictly Neumaier graph independently, using completely different approaches. Goryainov and Panasenko were looking for strictly Neumaier graphs that admit a partition into regular cliques and used this pattern for computer searching. The author found the graph in a collection of vertex-transitive edge-regular graphs received from Gordon Royle. Holt and Royle have recently enumerated all transitive permutation groups of degree at most 47 [42]. From this, Royle was able to enumerate all vertex-transitive edge-regular graphs on at most 47 vertices.

Thus we also find all vertex-transitive strictly Neumaier graphs on at most 47 vertices using the enumeration of Holt and Royle [42]. We list the parameters of all vertex-transitive strictly Neumaier graphs on at most 47 vertices, and the number of vertex-transitive strictly Neumaier graphs with these parameters. In the figures below, we give the adjacency matrices of all vertex-transitive strictly

Neumaier graphs with these parameters. In each of the matrices, we highlight at least one regular clique in the graph. The Appendix 4.A contains these graphs in GRAPE format, and information on the automorphism group of each graph.

(i) 1 graph with parameters $(16, 9, 4; 2, 4)$ (Figure 4.4).

(ii) 4 graphs with parameters$(24, 8, 2; 1, 4)$ (Figures 4.5 and 4.6). We note that the four vertex-transitive strictly Neumaier graphs in $\mathrm{NG}(24, 8, 2; 1, 4)$ appear in Goryainov and Shalaginov [35]. They come about in a search for Deza graphs, which are a certain generalisation of strongly regular graphs.

(iii) 2 graphs with parameters $(28, 9, 2; 1, 4)$ (Figure 4.7).

(iv) 1 graph with parameters $(40, 12, 2; 1, 4)$ (Figure 4.8).

$$
\begin{array}{l}
\boxed{0111}\,110010101001 \\
\boxed{1011}\,001101101100 \\
\boxed{1101}\,001110010011 \\
\boxed{1110}\,110001010110 \\
1001\,\boxed{0111}\,10010011 \\
1001\,\boxed{1011}\,01101100 \\
0110\,\boxed{1101}\,10101001 \\
0110\,\boxed{1110}\,01010110 \\
10101010\,\boxed{0111}\,1010 \\
01010101\,\boxed{1011}\,1010 \\
11000110\,\boxed{1101}\,0101 \\
00111001\,\boxed{1110}\,0101 \\
110001101100\,\boxed{0111} \\
010101010011\,\boxed{1011} \\
001110011100\,\boxed{1101} \\
101010100011\,\boxed{1110}
\end{array}
$$

Figure 4.4: The adjacency matrix of the vertex-transitive strictly Neumaier graph in $\mathrm{NG}(16, 9, 4; 2, 4)$.

0111 00101000100000100001	0111 10000100010000010010
1011 10000010010001000010	1011 01001000001001000100
1101 01000100000110000100	1101 00100010100010000001
1110 00010001001000011000	1110 00010001000100101000
0100 0111 0100010001000100	1000 0111 0001000100010010
0010 1011 0010000110000010	0100 1011 1000100001000001
1000 1101 1000001000010001	0010 1101 0010001010000100
0001 1110 0001100000101000	0001 1110 0100010000101000
10000010 0111 100000011000	01000100 0111 001010000001
00101000 1011 010010000100	10000001 1011 010000011000
01000100 1101 000101000010	00100010 1101 100001000100
00010001 1110 001000100001	00011000 1110 000100100010
100000011000 0111 00101000	001001000010 0111 01000001
010010000100 1011 10000010	100000010100 1011 00100010
000100100001 1101 00010001	010000101000 1101 10000100
001001000010 1110 01000100	000110000001 1110 00011000
0010010001000100 0111 0010	0010001010000010 0111 0001
0100100000100001 1011 0100	0100010000101000 1011 0100
1000000100011000 1101 0001	0001000100010100 1101 0010
0001001010000010 1110 1000	1000100001000001 1110 1000
00010001100010000001 0111	00010001010000010001 0111
00101000010000010100 1011	01000010001000100100 1011
01000100001001001000 1101	10001000000101000010 1101
10000010000100100010 1110	00100100100010001000 1110

Figure 4.5: The adjacency matrices of two of the vertex-transitive strictly Neumaier graphs in NG$(24, 8, 2; 1, 4)$.

0111 10001000001000010010	0111 10001000001001000010
1011 00100100010000100100	1011 00100100010000010100
1101 01000010100010001000	1101 01000010100010000001
1110 00010001000101000001	1110 00010001000100101000
1000 0111 0001001001000010	1000 0111 0001001001001000
0010 1011 0100100010000100	0010 1011 0100100010000100
0100 1101 0010010000101000	0100 1101 0010010000010001
0001 1110 1000000100010001	0001 1110 1000000100100010
10000001 0111 001000010001	10000001 0111 000101000010
01000100 1011 100000100100	01000100 1011 100000010100
00100010 1101 010010001000	00100010 1101 010010000001
00011000 1110 000101000010	00011000 1110 001000101000
001001000100 0111 00101000	001001000100 0111 00010001
010000100010 1011 10000100	010000100010 1011 10000100
100010001000 1101 01000001	100010000001 1101 00100010
000100010001 1110 00010010	000100011000 1110 01001000
0010010000100100 0111 0100	0010010000100100 0111 0100
0001100000010010 1011 0001	1000100010000001 1011 1000
0100001001001000 1101 1000	0001000100010010 1101 0010
1000000110000001 1110 0010	0100001001001000 1110 0001
00100010001010000010 0111	00011000000100010100 0111
01000100010001001000 1011	01000100010001001000 1011
10001000000100010001 1101	10000001100000100010 1101
00010001100000100100 1110	00100010001010000001 1110

Figure 4.6: The adjacency matrices of two more of the vertex-transitive strictly Neumaier graphs in NG$(24, 8, 2; 1, 4)$.

0111 100010000001010001000100	0111 100010000010100010000100
1011 001000011000000100010001	1011 010001000001001000010001
1101 010000100100001000101000	1101 001000101000000101000010
1110 000101000010100010000010	1110 000100010100010000101000
1000 0111 10000010100001001000	1000 0111 00100100100000100100
0010 1011 00101000000100100010	0100 1011 00011000001001000001
0100 1101 00010100001000010100	0010 1101 10000001000100010010
0001 1110 01000001010010000001	0001 1110 01000010010010001000
10001000 0111 0100010000011000	10000010 0111 0001100010000010
00010001 1011 1000100000100001	01000001 1011 0010001000011000
00100100 1101 0010001001000010	00101000 1101 0100000101000100
01000010 1110 0001000110000100	00010100 1110 1000010000100001
010001000100 0111 000100100001	001001000001 0111 010001000010
001000101000 1011 001000011000	000110000010 1011 000100101000
000110000010 1101 100001000010	100000010100 1101 001010000100
100000010001 1110 010010000100	010000101000 1110 100000010001
0001100001000010 0111 00101000	1000100010000001 0111 00100001
1000000110000001 1011 00010001	0001000100011000 1011 10000010
0010001000100100 1101 01000100	0100010001000010 1101 01000100
0100010000011000 1110 10000010	0010001000100100 1110 00011000
00010001000100010001 0111 0010	10000001100000100100 0111 0010
10001000001000100010 1011 0100	00100100001010000010 1011 0100
00100100010010001000 1101 1000	00011000000101001000 1101 0001
01000010100001000100 1110 0001	01000010010000010001 1110 1000
001010001000010010000010 0111	000100010100010000010001 0111
100000100001000100100100 1011	100010000010001000100100 1011
000101000010001000011000 1101	001000101000100001001000 1101
010000010100100001000001 1110	010001000001000110000010 1110

Figure 4.7: The adjacency matrices of the two vertex-transitive strictly Neumaier graphs in NG$(28, 9, 2; 1, 4)$.

0111 100010001000100010000100001010000010
1011 001000100010000100101000010000010100
1101 000100010001001000010010100000100001
1110 010001000100010001000001000101001000
1000 0111 00101000000101000010000100100010
0001 1011 00010100001010001000001000011000
0100 1101 10000010100000001000110000100010 0
0010 1110 01000001010000100100010010000001
10000010 0111 0100000100010001001010000001
00010001 1011 1000001000100100000101000100
01001000 1101 0001100001000010010000011000
00100100 1110 0010010010001000100000100010
100010000100 0111 100000100001100000100100
000101001000 1011 010000010100010000010001
010000100001 1101 000110000001000010100001 0
001000010010 1110 001001001000001010001000
1000001000101000 0111 00010010001000010100
0001000100010100 1011 00101000000100100001
0010010001000001 1101 10000100100001001000
0100100010000010 1110 01000001010010000010
10000100000100100010 0111 0100000110000100
00011000001000010001 1011 0001001001000001
01000001010010000100 1101 1000100000010010
00100010100001001000 1110 0010010000101000
010001000000100010100001 0 0111 001010000100
100000010100010000101000 1011 010000010010
001010000001000101000000 1 1101 000101000001
000100101000100000010100 1110 100000101000
0010001000011000001000100001 0111 01000010
0100000100100100000100010100 1011 10001000
1000010010000000110000100100 0 1101 00010001
0001100000100001001001000001 0 1110 00100100
10000001100000001000110001000010 0 0111 0100
000100100100000100010010000101000 1011 0001
001010000000110000100000010001000 1 1101 1000
010001000001001001000000100100001 0 1110 0010
000101000010000100100001000101000010 0111
010000100100100010001000100000011000 1011
100010000000100100001001001001000000 1 1101
001000011000010001000100001000100100 1110

Figure 4.8: The adjacency matrix of the vertex-transitive strictly Neumaier graph in NG$(40, 12, 2; 1, 4)$.

4.4.2 Non-vertex-transitive strictly Neumaier graphs

After the publication of [31], discussions between Soicher, Goryainov and the author prompted Soicher to suggest a certain computational approach to constructing Neumaier graphs. From this approach, he and the author have found several more examples of strictly Neumaier graphs, each of which is non-transitive. Soicher and the author are currently working on extending this approach.

We list the parameters of all non-vertex-transitive strictly Neumaier graphs we have found so far using this approach. In the figures below, we give the adjacency matrices of these graphs. In each of the matrices, we highlight at least one regular clique in the graph. Appendix 4.A contains these graphs in GRAPE format.

(i) 2 graphs with parameters $(24, 8, 2; 1, 4)$ (Figure 4.9). We note that these graphs each contain a unique regular clique, and the second graph has diameter 3. These are both previously unseen properties of strictly Neumaier graphs.

    Determining the existence of a diameter 3 strictly Neumaier graph was an interesting open problem before the discovery of this graph. The diameter of a Neumaier graph is at most 3, as any vertex not in a fixed regular clique is adjacent to at least one vertex of the clique. Therefore, the possible diameters of a strictly Neumaier graph are 2 and 3.

(ii) 2 graphs with parameters $(28, 9, 2; 1, 4)$ (Figure 4.10). We note that for both of these graphs, each vertex lies in at least 1 regular clique, but there is no spread of cliques (a partition of the vertices into regular cliques). This is another property which was not previously observed in strictly Neumaier graphs.

010011100001000010011000
101000110000100101001000
010100011000010010101000
001010001100001001011000
100101000110000100101000
100010011010010000010100
110000001101001100000100
011001000110100010000100
001101100001010001000100
000110110000101000100100
000011010001001110000010
100000101010100011000010
010000010101010001100010
001001001000101000110010
000100100110010100010010
010010100010001001100001
101000010011000000110001
010100001001100100010001
001010000100110110000001
100101000000011011000001
11111000000000000000 0111
00000111110000000000 1011
00000000001111100000 1101
00000000000000011111 1110

010011100001000010011000
101000110000100101001000
010100011000010010101000
001010001100001001011000
100101000110000100101000
100010011010010000010100
110000001101001100000100
011001000110100010000100
001101100001010001000100
000110110000101000100100
000011010001001110000010
100000101010100011000010
010000010101010001100010
001001001000101000110010
000100100110010100010010
010010100010001001100001
101000010011000000110001
010100001001100100010001
001010000100110110000001
100101000000011011000001
11111000000000000000 0111
00000111110000000000 1011
00000000001111100000 1101
00000000000000011111 1110

Figure 4.9: The adjacency matrices of the known non-vertex-transitive strictly Neumaier graphs in NG$(24, 8, 2; 1, 4)$.

0110011000010000011000101000	0110100010010000010100011000
1011000101000000100100011000	1010011001000000100011001000
1100100010100001000011001000	1101000100100001001000101000
0100110011000010000101000100	0010110100010010000100100100
0011011000101000000010100100	1001010011001000000010010100
1001100100010100001000010100	0101101000100100001001000100
1000100110011000100000100010	0100010110100010100000100010
0100011011000101000000010010	0011001010011001000000010010
0011001100100010010001000010	1000101101000100010001000010
0101000100110011000010000001	0100100010110100100010000001
0010100011011000011000000001	0010011001010011001000000001
1000011001100100100100000001	1001000101101000010100000001
0000101000100110101000011000	0000100100010110010010011000
0000010100011011000101001000	0000010011001010101001001000
0001000011001100010010101000	0001001000101101000100101000
0010000101000100110011000100	0010000100100010110100010100
0100001000011001010100010100	0100001001000101011000100100
1000000010100011101000100100	1000000010011001100011000100
1000010000101000010110010010	0010010000100100100110100010
0101000000010100101011000010	1001000000010011001010010010
0010100001000011001100100010	0100100001001000011101000010
0011000010000101000100110001	0100010010000100010010110001
1000101000000010010011010001	0011001000000010101001010001
0100010100001000101001100001	1000100100001001000101100001
111000000000111000000000 0111	111000000000111000000000 0111
000111000000000111000000 1011	000111000000000111000000 1011
000000111000000000111000 1101	000000111000000000111000 1101
000000000111000000000111 1110	000000000111000000000111 1110

Figure 4.10: The adjacency matrices of the known non-vertex-transitive strictly Neumaier graphs in NG$(28, 9, 2; 1, 4)$.

4.A Small strictly Neumaier graphs in GAP

This appendix contains the graphs with adjacency matrices found in Sections 4.4.1 and 4.4.2, in GRAPE format.

GRAPE [55] is a distributed package of the computational algebra system GAP, and is mainly used to analyse finite simple graphs. In particular, the functionality of our new package AGT (see Appendix A) is based on graphs stored in GRAPE format. In GRAPE format, a graph `gamma` is stored as a record, with mandatory components `isGraph`, `order`, `group`, `schreierVector`, `representatives`, and `adjacencies`. Here we give a brief description of the GRAPE graph format, which is discussed further in the package manual for GRAPE [55].

The `order` component contains the number of vertices of `gamma`. The vertices of gamma are always `1,2,...,gamma.order` (but may be given names by a user or a function). The `group` component records the GAP permutation group associated with `gamma`, and must be a subgroup of the automorphism group of `gamma`. The `representatives` component records a set of orbit representatives for the action of `gamma.group` on the vertices of gamma, with `gamma.adjacencies[i]` being the set of vertices adjacent to `gamma.representatives[i]`.

In the graphs below, we give the graphs with adjacency matrices found in the Figures 4.4 to 4.10, in GRAPE format. For each of the graphs, the `group` component has been set as the full automorphism group of the graph.

1. The vertex-transitive strictly Neumaier graph in $\mathrm{NG}(16, 9, 4; 2, 4)$ (Figure 4.4). The automorphism group of this graph has order 256.

```
gap> rec(adjacencies := [ [ 2, 3, 4, 5, 6, 9, 10, 14, 15 ] ],
group := Group( [
( 9,14)(11,16), (10,15)(12,13),
( 2, 3)( 5, 6)( 9,10)(11,12)(13,16)(14,15),
( 2, 5)( 3, 6)( 9,10)(11,12)(13,16)(14,15),
( 1, 2)( 3, 4)( 5, 7)( 6, 8)( 9,11)(14,16),
( 1, 9, 3,12, 4,11, 2,10)( 5,13, 7,16, 6,15, 8,14) ] ),
isGraph := true,order := 16,representatives := [ 1 ],
schreierVector := [ -1, 5, 3, 5, 4, 4, 5, 5, 6, 6, 5, 6, 6, 1, 2,
 5 ] );
```

2. The vertex-transitive strictly Neumaier graphs in NG$(24, 8, 2; 1, 4)$ (Figures 4.5 and 4.6). With respect to the order in which the graphs are presented below, the automorphism group of each graph has order 480, 96, 96 and 480.

```
gap> rec(adjacencies := [ [ 2, 3, 4, 7, 9, 13, 19, 24 ] ],
group := Group( [
(9,24)(10,22)(11,23)(12,21)(13,19)(14,18)(15,20)(16,17),
( 5,11)( 6,10)( 7, 9)( 8,12)(13,24)(14,23)(15,21)(16,22),
( 2, 3)( 5, 6)(10,11)(14,16)(17,18)(22,23),
( 1, 2)( 3, 4)( 5, 7)( 6, 8)( 9,14,24,18)(10,15,22,20)(11,13,23,19)
(12,16,21,17),
( 1, 5)( 2, 7)( 3, 8)( 4, 6)( 9,14)(10,13)(11,15)(12,16)(17,21)
(18,24)(19,22)(20,23) ] ),
isGraph := true,order := 24,representatives := [ 1 ],
schreierVector := [ -1, 4, 3, 4, 5, 3, 5, 5, 2, 3, 2, 2, 4, 4, 5,
4, 4, 4, 4, 5, 2, 3, 1, 1 ] );

gap> rec(adjacencies := [ [ 2, 3, 4, 5, 10, 14, 20, 23 ] ],
group := Group( [
( 5,20)( 6,18)( 7,17)( 8,19)( 9,22)(10,23)(11,24)(12,21),
( 2, 3)( 6, 7)( 9,11)(13,15)(17,18)(22,24),
( 1, 2)( 3, 4)( 5, 6)( 7, 8)( 9,23)(10,22)(11,21)(12,24)(13,16)
(14,15)(17,19)(18,20),
( 1, 5,12, 4, 8,10)( 2, 6, 9)( 3, 7,11)(13,17,22)
(14,20,23,16,19,21)(15,18,24) ] ),
isGraph := true,order := 24,representatives := [ 1 ],
schreierVector := [ -1, 3, 2, 3, 4, 4, 4, 4, 4, 1, 4, 4, 4, 4, 4,
4, 1, 1,1, 1, 1, 1, 4, 3 ] );

gap> rec(adjacencies := [ [ 2, 3, 4, 5, 9, 15, 20, 23 ] ],
group := Group( [
( 5,20)( 6,17)( 7,19)( 8,18)( 9,15)(10,14)(11,13)(12,16),
( 2, 3)( 6, 7)(10,11)(13,14)(17,19)(21,22),
( 1, 2)( 3, 4)( 5,10)( 6,12)( 7, 9)( 8,11)(13,18)(14,20)(15,19)
(16,17)(21,24)(22,23),
( 1, 5,12, 4, 8, 9)( 2, 6,11)( 3, 7,10)(13,21,19)(14,22,17)
(15,23,18,16,24,20) ] ),
isGraph := true,order := 24,representatives := [ 1 ],
schreierVector := [ -1, 3, 2, 3, 4, 4, 4, 4, 3, 3, 4, 4, 1, 3, 4,
1, 1, 1,1, 1, 4, 4, 4, 4 ] );
```

```
gap> rec(adjacencies := [ [ 2, 3, 4, 5, 9, 15, 18, 23 ] ],
group := Group( [
( 9,23)(10,22)(11,24)(12,21)(13,17)(14,20)(15,18)(16,19),
( 5, 9,15,18,23)( 6,11,13,17,24)( 7,10,14,20,22)( 8,12,16,19,21),
( 2, 3)( 6, 7)(10,11)(13,14)(17,20)(22,24),
( 1, 2)( 3, 4)( 5, 7)( 6, 8)( 9,10)(11,12)(13,16)(14,15)(17,19)
(18,20)(21,24)(22,23),
( 1, 5)( 2, 6)( 3, 7)( 4, 8)( 9,21)(10,22)(11,24)(12,23)(13,14)
(17,20) ] ),
isGraph := true,order := 24,representatives := [ 1 ],
schreierVector := [ -1, 4, 3, 4, 5, 5, 4, 4, 2, 4, 2, 4, 2, 4, 2,
4, 1, 1,4, 4, 5, 4, 1, 1 ] );
```

3. The vertex-transitive strictly Neumaier graph in $\mathrm{NG}(28, 9, 2; 1, 4)$ (Figure 4.7). With respect to the order in which the graphs are presented below, the automorphism group of each graph has order 56 and 168.

```
gap> rec(adjacencies := [ [ 2, 3, 4, 5, 9, 16, 18, 22, 26 ] ],
group := Group( [
( 2, 3)( 5,16)( 6,13)( 7,14)( 8,15)( 9,26)(10,27)(11,28)(12,25)
(17,21)(18,22)(19,24)(20,23),
( 1, 2)( 3, 4)( 5,13)( 6,15)( 7,16)( 8,14)( 9,28)(10,25)(11,27)
(12,26)(17,23)(18,24)(19,21)(20,22),
( 1, 5)( 2, 6)( 3, 7)( 4, 8)( 9,22)(10,21)(11,24)(12,23)(13,20)
(14,19)(15,18)(16,17)(25,26)(27,28) ] ),
isGraph := true,order := 28,representatives := [ 1 ],
schreierVector := [ -1, 2, 1, 2, 3, 3, 3, 3, 3, 3, 3, 3, 2, 1, 2,
 1, 3, 3,3, 3, 1, 2, 2, 2, 2, 2, 1, 2 ] );

gap> rec(adjacencies := [ [ 2, 3, 4, 5, 9, 15, 17, 21, 26 ] ],
group := Group( [
( 5,21)( 6,24)( 7,22)( 8,23)( 9,26)(10,28)(11,27)(12,25)(13,20)
(14,18)(15,17)(16,19),
( 2, 3, 4)( 5,15, 9)( 6,13,12)( 7,14,10)( 8,16,11)(17,26,21)
(18,28,22)(19,27,23)(20,25,24),
( 1, 2)( 3, 4)( 5, 6)( 7, 8)( 9,10)(11,12)(13,14)(15,16)(17,19)
(18,20)(21,24)(22,23)(25,27)(26,28),
( 1, 5)( 2, 6)( 3, 7)( 4, 8)( 9,11)(10,12)(13,24)(14,21)(15,23)
(16,22)(17,26)(18,25)(19,28)(20,27) ] ),
isGraph := true,order := 28,representatives := [ 1 ],
```

```
schreierVector := [ -1, 3, 2, 2, 4, 4, 4, 4, 2, 2, 4, 2, 2, 4, 2,
3, 2, 1,3, 2, 1, 1, 4, 1, 2, 2, 4, 3 ] );
```

4. The vertex-transitive strictly Neumaier graph in $\mathrm{NG}(40, 12, 2; 1, 4)$ (Figure 4.8). The automorphism group of this graph has order 240.

```
gap> rec(adjacencies := [ [ 2, 3, 4, 5, 9, 13, 17, 21, 26, 31, 33,
39 ] ],
group := Group( [
( 5, 9)( 6,10)( 7,11)( 8,12)(13,31)(14,32)(15,30)(16,29)(21,26)
(22,28)(23,25)(24,27)(33,39)(34,37)(35,40)(36,38),
( 5,13)( 6,14)( 7,15)( 8,16)( 9,21)(10,22)(11,23)(12,24)(17,39)
(18,37)(19,40)(20,38)(25,30)(26,31)(27,29)(28,32),
( 1, 2)( 3, 4)( 5, 7)( 6, 8)( 9,11)(10,12)(13,15)(14,16)(17,20)
(18,19)(21,23)(22,24)(25,26)(27,28)(29,32)(30,31)(33,36)(34,35)
(37,40)(38,39),
( 1, 3)( 2, 4)( 5, 8)( 6, 7)( 9,12)(10,11)(13,16)(14,15)(17,19)
(18,20)(21,24)(22,23)(25,28)(26,27)(29,31)(30,32)(33,35)(34,36)
(37,38)(39,40),
( 1, 5)( 2, 7)( 3, 8)( 4, 6)( 9,11)(10,12)(13,39)(14,37)(15,38)
(16,40)(17,20)(18,19)(21,32)(22,31)(23,29)(24,30)(25,34)(26,35)
(27,33)(28,36) ] ),
isGraph := true,order := 40,representatives := [ 1 ],
schreierVector := [ -1, 3, 4, 4, 5, 5, 5, 5, 1, 1, 1, 1, 2, 2, 2,
2, 2, 2,2, 2, 2, 5, 2, 2, 1, 1, 1, 2, 1, 1, 1, 5, 1, 1, 1, 1, 5, 5,
5, 5 ] );
```

5. The known non-vertex-transitive strictly Neumaier graphs in $\mathrm{NG}(24, 8, 2; 1, 4)$ (Figure 4.9). With respect to the order in which the graphs are presented below, the automorphism group of each graph has order 20 and 20.

```
gap> rec(adjacencies := [ [ 2, 5, 6, 7, 12, 17, 20, 21 ],
[ 1, 2, 3, 4, 5, 22, 23, 24 ] ],
group := Group([
( 1, 2, 3, 4, 5)( 6, 7, 8, 9,10)(11,12,13,14,15)(16,17,18,19,20),
( 1, 6,11,17)( 2, 9,15,19)( 3, 7,14,16)( 4,10,13,18)( 5, 8,12,20)
(21,22,23,24)]),
isGraph := true,isSimple := true, order := 24,
representatives := [ 1, 21 ],
```

```
schreierVector := [ -1, 1, 1, 1, 1, 2, 1, 1, 2, 1, 2, 1, 2, 2, 2,
2, 2, 1, 2, 2, -2, 2, 2, 2 ] );

gap> rec(adjacencies := [ [ 2, 5, 6, 7, 12, 17, 20, 21 ],
[ 1, 2, 3, 4, 5, 22, 23, 24 ] ],
group := Group([
( 1, 2, 3, 4, 5)( 6, 7, 8, 9,10)(11,12,13,14,15)(16,17,18,19,20),
( 1, 6,15,20)( 2, 8,14,18)( 3,10,13,16)( 4, 7,12,19)( 5, 9,11,17)
(21,22,23,24) ]),
isGraph := true,isSimple := true, order := 24,
representatives := [ 1, 21 ],
schreierVector := [ -1, 1, 1, 1, 1, 2, 1, 2, 1, 2, 1, 2, 2, 2, 2,
1, 2, 2, 2, 2, -2, 2, 2, 2 ] );
```

6. The known non-vertex-transitive strictly Neumaier graphs in $\mathrm{NG}(28, 9, 2; 1, 4)$ (Figure 4.10). With respect to the order in which the graphs are presented below, the automorphism group of each graph has order 96 and 96.

```
gap> rec(adjacencies :=[ [ 2, 3, 6, 7, 12, 18, 19, 23, 25 ],
 [ 1, 2, 3, 13, 14, 15, 26, 27, 28 ] ],
group := Group( [
( 2, 3)( 4,16)( 5,17)( 6,18)( 8, 9)(10,22)(11,24)(12,23)(14,15)
(20,21),
( 4,10)( 5,11)( 6,12)( 7,19)( 8,20)( 9,21)(16,22)(17,24)(18,23)
(26,28),
( 1, 2, 3)( 4, 5, 6)( 7, 8, 9)(10,11,12)(13,14,15)(16,18,17)
(19,20,21)(22,23,24),
( 1, 4,19,22,13,16, 7,10)( 2, 6,20,24,14,17, 8,12)
( 3, 5,21,23,15,18, 9,11)(25,26,27,28) ] ),
isGraph := true,order := 28,representatives := [ 1, 25 ],
schreierVector := [ -1, 3, 1, 4, 3, 4, 4, 2, 4, 2, 3, 2, 4, 4, 4,
1, 1, 1,4, 4, 4, 2, 2, 4, -2, 4, 4, 2 ] );

gap> rec(adjacencies :=[ [ 2, 3, 5, 9, 12, 18, 20, 24, 25 ],
 [ 1, 2, 3, 13, 14, 15, 26, 27, 28 ] ],
group := Group( [
( 2, 3)( 4,10)( 5,12)( 6,11)( 7,19)( 8,21)( 9,20)(14,15)(16,22)
(17,23)(18,24)(26,28),
( 4,16)( 5,18)( 6,17)(10,22)(11,23)(12,24),
( 1, 2, 3)( 4, 5, 6)( 7, 8, 9)(10,11,12)(13,14,15)(16,18,17)
```

```
(19,20,21)(22,23,24),
( 1, 4)( 2, 6)( 3, 5)( 7,22)( 8,24)( 9,23)(10,19)(11,21)(12,20)
(13,16)(14,17)(15,18)(25,26)(27,28) ] ),
isGraph := true,order := 28,representatives := [ 1, 25 ],
schreierVector := [ -1, 3, 1, 4, 3, 4, 4, 1, 4, 1, 1, 1, 4, 4, 4,
 2, 2, 3,4, 3, 4, 2, 2, 1, -2, 4, 4, 1 ] );
```

Chapter 5

Families of strictly Neumaier graphs

After discovering the smallest strictly Neumaier graph, we had answered the questions posed by Greaves and Koolen [37] (see Chapter 4) by giving a single example of a strictly Neumaier graph containing a 2-regular clique and having 16 vertices. However, it was still unknown whether this graph was a sporadic example of a graph with a regular clique with nexus not 1.

When studying the structure of the two known constructions of strictly Neumaier graphs of Greaves and Koolen [36, 37], we observed a generalisation which encompasses both constructions. Furthermore, we observed that each of the vertex-transitive strictly Neumaier graphs with parameters $(24, 8, 2; 1, 4)$, $(28, 9, 2; 1, 4)$ and $(40, 12, 2; 1, 4)$ presented in Section 4.4.1 are instances of the generalisation. Although this is of interest, any graph obtained from our generalisation can only contain regular cliques with nexus 1.

Therefore, we concentrate on generalising the smallest strictly Neumaier graph. A short discussion between the author and Goryainov led to Goryainov finding a generalisation of the smallest strictly Neumaier graph, by using a switching operation on the graphs from a well-known family of strongly regular graphs. After some computational investigation and conversation, we found a second generalisation which is similar to the first.

In this chapter, we will cover the first constructions of strictly Neumaier graphs.

We will start by giving a generalisation of the constructions of Greaves and Koolen [37] and [36]. Then we present two new infinite sequences of strictly Neumaier graphs. Each of these sequences has first element the unique smallest strictly Neumaier graph. The i^{th} element of each of these sequences is a strictly Neumaier graph which contains a 2^i-regular clique. In fact, all of these graphs contain a subgraph isomorphic to a clique extension (see the definition of a clique extension in Brouwer, Cohen and Neumaier [14, p. 6]) of the unique smallest strictly Neumaier graph. These constructions show that the nexus of a clique in a strictly Neumaier graph is not bounded above by some constant number. Furthermore, each of the graphs in these sequences has the edge-regular graph parameters of an affine polar graph.

5.1 Families with fixed nexus 1

In [36, Theorem 2.1], Greaves and Koolen use antipodal classes of a distance-regular graph to construct (strictly) Neumaier graphs. In particular, they take copies of a distance-regular graph with parameters with certain properties, and a partition of its vertex set into antipodal classes. Then, they match these classes in a natural way and insert edges to create a Neumaier graph.

In our generalisation, we take the same approach but relax some of the conditions on the structure of the graphs. We replace several copies of a distance-regular graph with a sequence of (not necessarily isomorphic) edge-regular graphs with the same parameters, satisfying certain conditions. Then we replace the partitions into antipodal classes with partitions into perfect 1-codes. Finally, we also generalise how the perfect 1-codes in each graph are matched.

For a positive integer r, a *perfect r-code* in a graph Γ is a subset U of vertices such that for each vertex $w \in V(\Gamma)$, there is a unique vertex $u \in U$ such that $d(u, w) \leq r$.

Let Γ be a graph with perfect r-code U. Note that as $d(w, w) = 0$ for all vertices w, a perfect r-code is always an independent set. Also note that when $r = 1$ the set U is 1-regular, so U is a $(0, 1)$-regular set. From now on, we will only consider perfect 1-codes.

Let $\Gamma^{(1)}, \ldots, \Gamma^{(t)} \in \mathrm{ERG}(v, k, \lambda)$ such that each $\Gamma^{(i)}$ has a partition of their vertices into perfect 1-codes of size a, where a is a proper divisor of $\lambda + 2$. Further, suppose $t = (\lambda+2)/a$.

For any $j \in \{1, ..., t\}$, let $H_1^{(j)}, \dots, H_{v/a}^{(j)}$ denote the perfect 1-codes that partition the vertex set of $\Gamma^{(j)}$.

Let $\Pi = (\pi_2, \dots, \pi_t)$ be a $(t-1)$-tuple of permutations from $\mathrm{Sym}(\{1, ..., \frac{v}{a}\})$.

We now construct a new graph as follows:

1. Take the disjoint union of the graphs $\Gamma^{(1)}, \dots, \Gamma^{(t)}$.

2. For any $i \in \{1, ..., v/a\}$, connect any two vertices from $H_i^{(1)}, H_{\pi_2(i)}^{(2)}, \dots, H_{\pi_t(i)}^{(t)}$ to form a 1-regular clique of size ta.

Denote the resulting graph by $F_\Pi(\Gamma^{(1)}, \dots, \Gamma^{(t)})$. Then an argument similar to Greaves and Koolen [36, Theorem 2.1] shows that the resulting graphs are Neumaier graphs.

Theorem 5.1. *Let $\Gamma^{(1)}, \dots, \Gamma^{(t)} \in ERG(v, k, \lambda)$ such that each $\Gamma^{(j)}$ has a partition of their vertices into perfect 1-codes $H_1^{(j)}, \dots, H_{v/a}^{(j)}$, each of size a, where a is a proper divisor of $\lambda+2$ and $t = (\lambda+2)/a$.*

Let $\Pi = (\pi_2, \dots, \pi_t)$ be a $(t-1)$-tuple of permutations from Sym$(\{1, ..., v/a\})$. Then

1. *$F_\Pi(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ has (a spread of) 1-regular cliques, each of size $\lambda+2$;*

2. *$F_\Pi(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ is an edge-regular graph with parameters $(v(\lambda+2)/a, k+\lambda+1, \lambda)$;*

Proof. 1. For $j \in \{1, ..., t\}$, consider the union of the sets $H_i^{(1)}, H_{\pi_2(i)}^{(2)}, \dots, H_{\pi_t(i)}^{(t)}$. By construction, this is a clique in $F_\Pi(\Gamma^{(1)}, \dots, \Gamma^{(t)})$. Furthermore, as $H_i^{(j)}$ are perfect 1-codes in $\Gamma^{(i)}$, they are $(0,1)$-regular sets in $\Gamma^{(i)}$. Therefore, $H_i^{(1)}, H_{\pi_2(i)}^{(2)}, \dots, H_{\pi_t(i)}^{(t)}$ is a $(0,1)$-regular set in the disjoint union of the graphs $\Gamma^{(1)}, \dots, \Gamma^{(t)}$. After adding all possible edges in $H_i^{(1)}, H_{\pi_2(i)}^{(2)}, \dots, H_{\pi_t(i)}^{(t)}$, this set remains 1-regular.

2. The graph $F_\Pi(\Gamma^{(1)}, \dots, \Gamma^{(t)})$ has $vt = v(\lambda+2)/a$ vertices by definition.

Let w be a vertex in $F_\Pi(\Gamma^{(1)}, \dots, \Gamma^{(t)})$, so $w \in H_i^{(j)}$ for some i, j. Note that w is adjacent to k vertices of $\Gamma^{(j)}$ before adding the edges of the construction. Then w has $a-1$ new neighbours in $H_i^{(j)}$, and a new neighbours in $H_{\pi_m(\pi_j^{-1}(i))}^{(m)}$, for each $m \neq j$. Therefore, the degree of w is $k + a - 1 + a(t-1) = k + \lambda + 1$.

Now take a vertex u that is adjacent to w. Suppose u, w lie in the same clique $H_i^{(1)}, H_{\pi_2(i)}^{(2)}, \ldots, H_{\pi_t(i)}^{(t)}$, for some i. As this clique is 1-regular, u, w must have $at-2 = \lambda$ common neighbours. Otherwise, u, w must lie in different parts $H_m^{(j)}, H_n^{(j)}$ of the partition of a graph $\Gamma^{(j)}$, for some j. As these parts are perfect codes in $\Gamma^{(j)}$, there are no new common neighbours of u, w in the construction. □

When further knowledge of the structure or parameters of the graphs $\Gamma^{(j)}$ is assumed, we can prove that the resulting graph is not strongly regular. For example, in the original construction by Greaves and Koolen [36], the extra structure is that $\Gamma^{(j)}$ are isomorphic distance-regular graphs. Another approach is to use the conditions in Proposition 3.1 to prove the non-existence of a strongly regular graph with the Neumaier graph parameters of the resulting graph.

Now we give some examples of our construction which are not applications of the second construction of Greaves and Koolen [36].

Example 5.2. Let $\Gamma^{(1)}, \Gamma^{(2)}$ be identical copies of the icosohedral graph Γ. We know that Γ is in $\mathrm{ERG}(12, 5, 2)$, and is a 2-antipodal diameter 3 distance-regular graph. The vertices of this graph can be partitioned into antipodal pairs, and each antipodal pair is a perfect 1-code. Applying the original construction of Greaves and Koolen [36] to this partition, we find a strictly Neumaier graph in $\mathrm{NG}(24, 8, 2; 1, 4)$. By varying the antipodal classes and the permutation Π in our construction, we find 3 more non-isomorphic graphs. These four graphs are exactly the vertex-transitive strictly Neumaier graphs found in Goryainov and Shalaginov [35] referred to in Section 4.4.1. △

Example 5.3. Consider the Cayley graph found by Greaves and Koolen [37], which is a strictly Neumaier graph in $\mathrm{NG}(28, 9, 2; 1, 4)$. This graph is an instance of our generalisation, where we start with a single edge-regular graph with parameters $(28, 6, 2)$, coming from taking a quotient of the infinite triangular lattice. By taking a different quotient of the infinite triangular lattice, we can construct the other vertex-transitive strictly Neumaier graph with parameters $(28, 9, 2; 1, 4)$, found in Section 4.4.1. Recently we have used a quotient of the graph of the dodecahedral-icosahedral honeycomb to construct a strictly Neumaier graph with parameters $(78, 12, 4; 1, 6)$. We hope to extend these constructions by taking a family of lattices as the underlying graphs. △

Example 5.4. Consider two identical copies of the dodecahedral graph $\Delta^{(1)}, \Delta^{(2)}$. Now let Γ be the graph constructed as follows:

1. Take the disjoint union of the graphs $\Delta^{(1)}, \Delta^{(2)}$.
2. For all $w_1 \in V(\Delta^{(1)})$, take the corresponding vertex $w_2 \in V(\Delta^{(2)})$. Then add edges between w_1 and each vertex $u \in V(\Delta^{(2)})$ such that $d_{\Delta^{(2)}}(w_2, u) = 2$.
3. For all $w_2 \in V(\Delta^{(2)})$, take the corresponding vertex $w_1 \in V(\Delta^{(1)})$. Then add edges between w_2 and each vertex $u \in V(\Delta^{(1)})$ such that $d_{\Delta^{(1)}}(w_1, u) = 2$.

Then Γ is in $\mathrm{ERG}(40, 9, 2)$ and has a partition of its vertices into perfect 1-codes of size 4. Applying our construction with a single copy of Γ, we find the vertex-transitive strictly Neumaier graph with parameters $(40, 12, 2; 1, 4)$ found in Section 4.4.1. $\triangle$

5.2 Families with increasing nexus 2^i

In this section, we will construct two sequences of strictly Neumaier graphs that generalise the smallest strictly Neumaier graph. The motivation behind both constructions is as follows.

Consider a graph Γ, and two disjoint subsets S, T of the vertices of Γ. Now we introduce an important operation on the graph Γ. For each vertex u in S, we do the following. First take $N = \Gamma(u) \cap T$, the neighbours of u in T, and $M = T \setminus N$. Then delete all edges uw where w is in N, and insert all edges uw where w is in M. We will call this operation a *switching* of the edges between S and T in the graph Γ.

Note that the smallest Neumaier graph contains disjoint 2-regular 4-cliques. A switching between any distinct pair of these cliques will not change the fact that they are 2-regular. Therefore, if we could find a strongly regular graph with these parameters and containing disjoint 2-regular 4-cliques, we could hope that the smallest Neumaier graph is the result of switching edges between them. In general, we would like to find an infinite family of strongly regular graphs containing m-regular $2m$-cliques, and apply switchings between disjoint regular cliques to find strictly Neumaier graphs.

We will see that this approach is successful when applied to strongly regular graphs known as affine polar graphs.

5.2.1 Affine polar graphs $VO^+(2e, 2)$

Let V be a $(2e)$-dimensional vector space over a finite field $\mathbb{F}_q$, where $e \geq 2$ and q is a prime power, provided with the hyperbolic quadratic form $Q(x) = x_1x_2 + x_3x_4 + \ldots + x_{2e-1}x_{2e}$. The set Q^+ of zeroes of Q is called the *hyperbolic quadric*, where e is the maximal dimension of a subspace in Q^+. A *generator* of Q^+ is a subspace of maximal dimension e in Q^+.

Lemma 5.5. *Given an $(e-1)$-dimensional subspace W of Q^+, there are precisely two generators that contain W.*

Proof. This is a classical result related to polar spaces, the proof of which can be found in De Bruyn [26, Theorem 7.130]. □

Denote by $VO^+(2e, q)$ the graph on V with two vectors x, y being adjacent if and only if $Q(x - y) = 0$. The graph $VO^+(2e, q)$ is known as an *affine polar graph* (this is a well known family of strongly regular graphs, the properties of which can be found in Brouwer and Haemers [15], Brouwer and Shult [16] and De Bruyn [26]).

Lemma 5.6. *The graph $VO^+(2e, q)$ is a vertex-transitive strongly regular graph with parameters*

$$\begin{aligned} v &= q^{2e} \\ k &= (q^{e-1} + 1)(q^e - 1) \\ \lambda &= q(q^{e-2} + 1)(q^{e-1} - 1) + q - 2 \\ \mu &= q^{e-1}(q^{e-1} + 1). \end{aligned} \tag{5.1}$$

Note that $VO^+(2e, q)$ is isomorphic to the graph defined on the set of all $(2 \times e)$-matrices over $\mathbb{F}_q$ of the form

$$\begin{pmatrix} x_1 & x_3 & \ldots & x_{2e-1} \\ x_2 & x_4 & \ldots & x_{2e} \end{pmatrix}, \tag{5.2}$$

where two matrices are adjacent if and only if the scalar product of the first and the second rows of their difference is equal to 0.

Lemma 5.7. *There is a one-to-one correspondence between cosets of generators of Q^+ and maximal cliques in $VO^+(2e, q)$.*

Proof. Since $VO^+(2e, q)$ is vertex-transitive and the additive shift by an element is an automorphism of $VO^+(2e, q)$, it suffices to prove that every maximal clique containing the zero vector is a generator. The fact that every maximal clique containing the zero vector is a generator can be seen from results in the theory of polar spaces. For example, we can use the results found in De Bruyn [26, Theorem 7.3], [26, Corollary 7.16] and [26, Corollary 7.137]. □

Lemma 5.8. *Every maximal clique in $VO^+(2e, q)$ is a q^{e-1}-regular q^e-clique.*

Proof. This follows from Lemma 4.5, Lemma 4.6 and Brouwer, Cohen and Neumaier [14, Proposition 1.3.2(ii)]. □

A *spread* in $VO^+(2e, q)$ is a set of q^e disjoint maximal cliques that correspond to all cosets of a generator.

5.2.2 Construction 1

In the following subsections we will see that the smallest strictly Neumaier graph is the result of two consecutive switchings of the graph $VO^+(4, 2)$. We then generalise our switchings to the graphs $VO^+(2e, 2)$ for larger e, and construct an infinite sequence of strictly Neumaier graphs with the same edge-regular parameters as $VO^+(2e, 2)$. From now on, we will denote $VO^+(2e, 2)$ as the graph Γ_e. Throughout this section we use matrix notation with stars '*' as entries, which denotes the set of corresponding matrices where the stars take all possible values from $\mathbb{F}_2$.

The first construction of the smallest strictly Neumaier graph

Consider the 1-dimensional subspace

$$W = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

According to Lemma 5.5, the subspace W is contained in exactly two generators. These are the spaces

$$W_1 = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \text{ and } W_2 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Take the vector

$$u = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and consider the cosets

$$u + W_1 = \begin{pmatrix} * & * \\ 1 & 0 \end{pmatrix},$$

$$u + W_2 = \begin{pmatrix} * & 0 \\ 1 & * \end{pmatrix},$$

whose intersection is

$$u + W = \begin{pmatrix} * & 0 \\ 1 & 0 \end{pmatrix}.$$

In this setting, the adjacency matrix of the affine polar graph $\Gamma_2 = VO^+(4,2)$ can be seen in Table 5.1. The graph Γ_2 is isomorphic to the complement of the square lattice graph $L_2(4)$.

We note that switching edges between the cliques W_1, $u + W_1$ gives a graph isomorphic to the complement of the Shrikhande graph. The switching of edges between the cliques W_1, $u+W_1$ and then between the cliques W_2, $u+W_2$ is equivalent to inverting the highlighted entries in Table 5.1. This gives the strictly Neumaier graph $\Gamma_{2,1}$ on 16 vertices, whose adjacency matrix is presented in Table 5.2. The notation in the rightmost column of Table 5.2 means the following. Two rows have the same letter if and only if they correspond to non-adjacent vertices having 8 common neighbours; two rows have the same number if and only if they correspond to non-adjacent vertices having 4 common neighbours. Otherwise, every two non-adjacent vertices have 6 common neighbours; every two adjacent vertices have 4 common neighbours.

	01 00	11 00	00 00	10 00	00 01	10 01	01 10	11 10	00 10	10 10	00 11	10 11	01 01	11 01	01 11	11 11
01 00	0	1	1	1	0	0	1	0	1	0	0	1	1	1	1	0
11 00	1	0	1	1	0	0	0	1	0	1	1	0	1	1	0	1
00 00	1	1	0	1	1	1	1	0	1	0	1	0	0	0	0	1
10 00	1	1	1	0	1	1	0	1	0	1	0	1	0	0	1	0
00 01	0	0	1	1	0	1	0	1	1	0	1	0	1	1	1	0
10 01	0	0	1	1	1	0	1	0	0	1	0	1	1	1	0	1
01 10	1	0	1	0	0	1	0	1	1	1	0	0	1	0	1	1
11 10	0	1	0	1	1	0	1	0	1	1	0	0	0	1	1	1
00 10	1	0	1	0	1	0	1	1	0	1	1	1	0	1	0	0
10 10	0	1	0	1	0	1	1	1	1	0	1	1	1	0	0	0
00 11	0	1	1	0	1	0	0	0	1	1	0	1	1	0	1	1
10 11	1	0	0	1	0	1	0	0	1	1	1	0	0	1	1	1
01 01	1	1	0	0	1	1	1	0	0	1	1	0	0	1	1	0
11 01	1	1	0	0	1	1	0	1	1	0	0	1	1	0	0	1
01 11	1	0	0	1	1	0	1	1	0	0	1	1	1	0	0	1
11 11	0	1	1	0	0	1	1	1	0	0	1	1	0	1	1	0

Table 5.1: The adjacency matrix, A_2, of $\Gamma_2 = VO^+(4,2)$

The first generalisation of the smallest strictly Neumaier graph

In this subsection we generalise the construction above.

	01 00	11 00	00 00	10 00	00 01	10 01	01 10	11 10	00 10	10 10	00 11	10 11	01 01	11 01	01 11	11 11	
01 00	0	1	1	1	0	0	0	1	0	1	0	1	1	1	1	0	A1
11 00	1	0	1	1	0	0	1	0	1	0	1	0	1	1	0	1	B2
00 00	1	1	0	1	1	1	0	1	1	0	0	1	0	0	0	1	E5
10 00	1	1	1	0	1	1	1	0	0	1	1	0	0	0	1	0	F6
00 01	0	0	1	1	0	1	0	1	0	1	0	1	1	1	1	0	A2
10 01	0	0	1	1	1	0	1	0	1	0	1	0	1	1	0	1	B1
01 10	0	1	0	1	0	1	0	1	1	1	0	0	1	0	1	1	C3
11 10	1	0	1	0	1	0	1	0	1	1	0	0	0	1	1	1	D4
00 10	0	1	1	0	0	1	1	1	0	1	1	1	0	1	0	0	G7
10 10	1	0	0	1	1	0	1	1	1	0	1	1	1	0	0	0	H8
00 11	0	1	0	1	0	1	0	0	1	1	0	1	1	0	1	1	C4
10 11	1	0	1	0	1	0	0	0	1	1	1	0	0	1	1	1	D3
01 01	1	1	0	0	1	1	1	0	0	1	1	0	0	1	1	0	F5
11 01	1	1	0	0	1	1	0	1	1	0	0	1	1	0	0	1	E6
01 11	1	0	0	1	1	0	1	1	0	0	1	1	1	0	0	1	H7
11 11	0	1	1	0	0	1	1	1	0	0	1	1	0	1	1	0	G8

Table 5.2: The adjacency matrix, $A_{2,1}$, of the graph $\Gamma_{2,1}$

Take the $(e-1)$-dimensional subspace

$$W = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 0 & 0 \end{array} \right),$$

where the size of matrices is $2 \times e$. According to Lemma 5.5, the subspace W is contained in exactly two generators. These are the spaces

$$W_1 = \left(\begin{array}{ccc|cc} * & \dots & * & * & * \\ 0 & \dots & 0 & 0 & 0 \end{array}\right) \text{ and } W_2 = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 0 & * \end{array}\right).$$

Take the vector

$$u = \left(\begin{array}{ccc|cc} 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array}\right)$$

and consider the cosets

$$u + W_1 = \left(\begin{array}{ccc|cc} * & \dots & * & * & * \\ 0 & \dots & 0 & 1 & 0 \end{array}\right), \quad u + W_2 = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 1 & * \end{array}\right),$$

whose intersection is

$$u + W = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 1 & 0 \end{array}\right).$$

Denote by $\Gamma_{e,1} = \Gamma_e(W, W_1, W_2, u)$ the graph obtained from $\Gamma_e = VO^+(2e, 2)$ by switching edges between the cliques W_1, $u + W_1$ and then between the cliques W_2, $u + W_2$. Let (v, k, λ, μ) be the parameters of the affine polar graph $\Gamma_e = VO^+(2e, 2)$ as a strongly regular graph.

Theorem 5.9. *The graph $\Gamma_{e,1}$ is a strictly Neumaier graph with parameters*

$$(2^{2e}, (2^{e-1}+1)(2^e-1), 2(2^{e-2}+1)(2^{e-1}-1); 2^{e-1}, 2^e).$$

Further, the number of common neighbours of two non-adjacent vertices in the graph takes the values $\mu - 2^{e-1}, \mu$ and $\mu + 2^{e-1}$.

Proof. For any $a, b, c, d \in \mathbb{F}_2$, let

ab

cd

denote the set of matrices

$$\left(\begin{array}{ccc|cc} * & \dots & * & a & b \\ 0 & \dots & 0 & c & d \end{array}\right).$$

Consider the subgraph Δ of $\Gamma_e = VO^+(2e, 2)$ induced by the set of all matrices

$$\left(\begin{array}{ccc|cc} * & \dots & * & a & b \\ 0 & \dots & 0 & c & d \end{array}\right),$$

where a, b, c, d run over $\mathbb{F}_2$. The adjacency matrix of the subgraph Δ is presented by the block-matrix in Figure 3, where K denotes the adjacency matrix of the complete graph on 2^{e-2} vertices; J denotes the all-ones matrix of size $2^{e-2} \times 2^{e-2}$; Z denotes the all-zeroes matrix of size $2^{e-2} \times 2^{e-2}$.

We invert a block of the matrix by inverting each entry of the block. Therefore, inverting a block with entry J results in a block with entry Z, and inverting a block with entry Z results in a block with entry J. Switching edges between the cliques W_1, $u + W_1$ and then between the cliques W_2, $u + W_2$ is equivalent to inverting the highlighted entries in Table 5.3. This gives the submatrix of the adjacency matrix of $\Gamma_{e,1} = \Gamma(W, W_1, W_2, v)$ presented in Table 5.4. Note that every switched edge connects vertices from the subgraph Δ. This means that the switching preserves all edges having a vertex outside of Δ.

Let (v, k, λ, μ) be the parameters of the affine polar graph $\Gamma_e = VO^+(2e, 2)$ as a strongly regular graph. We have to check that the obtained graph is a strictly Neumaier graph. Note that W_1 is a regular clique in $\Gamma_{e,1} = \Gamma(W, W_1, W_2, u)$. Let us check that any pair of vertices in $\Gamma_{e,1}$ is OK, i.e., any two adjacent vertices have λ common neighbours. Also, we investigate which values of μ occur in $\Gamma_{e,1}$.

Let us consider any two vertices inside of Δ. The notation in the right column of the matrix in Table 5.4 means the following. Two block-rows have the same letter if and only if any row from the one block-row and any row from the other block-row correspond to non-adjacent vertices having $\mu + 2^{e-1}$ common neighbours; two block-rows have the same number if and only if any row from the one block-row and any row from the other block-row correspond to non-adjacent vertices having $\mu - 2^{e-1}$ common neighbours. Otherwise, every two non-adjacent vertices corresponding to rows of this submatrix have μ common neighbours; every two adjacent vertices have λ common neighbours. This means that all pairs of vertices inside of Δ are OK.

Let us consider any two vertices outside of Δ. Their neighbours and, consequently, their common neighbours are preserved by the switching. This means that all pairs of vertices outside of Δ are OK.

Let us consider a vertex x in Δ and a vertex y outside of Δ. If the neighbours

	01 00	11 00	00 00	10 00	00 01	10 01	01 10	11 10	00 10	10 10	00 11	10 11	01 01	11 01	01 11	11 11
01 00	K	J	J	J	Z	Z	J	Z	J	Z	Z	J	J	J	J	Z
11 00	J	K	J	J	Z	Z	Z	J	Z	J	J	Z	J	J	Z	J
00 00	J	J	K	J	J	J	J	Z	J	Z	J	Z	Z	Z	Z	J
10 00	J	J	J	K	J	J	Z	J	Z	J	Z	J	Z	Z	J	Z
00 01	Z	Z	J	J	K	J	Z	J	J	Z	J	Z	J	J	J	Z
10 01	Z	Z	J	J	J	K	J	Z	Z	J	Z	J	J	J	Z	J
01 10	J	Z	J	Z	Z	J	K	J	J	J	Z	Z	J	Z	J	J
11 10	Z	J	Z	J	J	Z	J	K	J	J	Z	Z	Z	J	J	J
00 10	J	Z	J	Z	J	Z	J	J	K	J	J	J	Z	J	Z	Z
10 10	Z	J	Z	J	Z	J	J	J	J	K	J	J	J	Z	Z	Z
00 11	Z	J	J	Z	J	Z	Z	Z	J	J	K	J	J	Z	J	J
10 11	J	Z	Z	J	Z	J	Z	Z	J	J	J	K	Z	J	J	J
01 01	J	J	Z	Z	J	J	J	Z	Z	J	J	Z	K	J	J	Z
11 01	J	J	Z	Z	J	J	Z	J	J	Z	Z	J	J	K	Z	J
01 11	J	Z	Z	J	J	Z	J	J	Z	Z	J	J	J	Z	K	J
11 11	Z	J	J	Z	Z	J	J	J	Z	Z	J	J	Z	J	J	K

Table 5.3: The adjacency matrix, A_e, of the subgraph Δ of $\Gamma_e = VO^+(2e, 2)$

of x are preserved by the switching, then x,y are OK. Assume that the neighbours of x are switched. Then the vertices x,y are OK since the vertex y is adjacent to

	01 00	**11 00**	**00 00**	**10 00**	**00 01**	**10 01**	**01 10**	**11 10**	**00 10**	**10 10**	**00 11**	**10 11**	**01 01**	**11 01**	**01 11**	**11 11**	
01 00	K	J	J	J	Z	Z	Z	J	Z	J	Z	J	J	J	J	Z	A1
11 00	J	K	J	J	Z	Z	J	Z	J	Z	J	Z	J	J	Z	J	B2
00 00	J	J	K	J	J	J	Z	J	J	Z	Z	J	Z	Z	Z	J	E5
10 00	J	J	J	K	J	J	J	Z	Z	J	J	Z	Z	Z	J	Z	F6
00 01	Z	Z	J	J	K	J	Z	J	Z	J	Z	J	J	J	J	Z	A2
10 01	Z	Z	J	J	J	K	J	Z	J	Z	J	Z	J	J	Z	J	B1
01 10	Z	J	Z	J	Z	J	K	J	J	J	Z	Z	J	Z	J	J	C3
11 10	J	Z	J	Z	J	Z	J	K	J	J	Z	Z	Z	J	J	J	D4
00 10	Z	J	J	Z	Z	J	J	J	K	J	J	J	Z	J	Z	Z	G7
10 10	J	Z	Z	J	J	Z	J	J	J	K	J	J	J	Z	Z	Z	H8
00 11	Z	J	Z	J	Z	J	Z	Z	J	J	K	J	J	Z	J	J	C4
10 11	J	Z	J	Z	J	Z	Z	Z	J	J	J	K	Z	J	J	J	D3
01 01	J	J	Z	Z	J	J	J	Z	Z	J	J	Z	K	J	J	Z	F5
11 01	J	J	Z	Z	J	J	Z	J	J	Z	Z	J	J	K	Z	J	E6
01 11	J	Z	Z	J	J	Z	J	J	Z	Z	J	J	J	Z	K	J	H7
11 11	Z	J	J	Z	Z	J	J	J	Z	Z	J	J	Z	J	J	K	G8

Table 5.4: The adjacency matrix, $A_{e,1}$, of the subgraph Δ of $\Gamma_{e,1} = \Gamma_e(W, W_1, W_2, u)$

half the of vertices of each block of Δ. In fact, the vertex y is given by a matrix

$$\left(\begin{array}{ccc|cc} y_1 & \cdots & y_{2e-5} & y_{2e-3} & y_{2e-1} \\ y_2 & \cdots & y_{2e-4} & y_{2e-2} & y_{2e} \end{array}\right),$$

where there is at least one non-zero among $y_2, y_4, \dots, y_{2e-4}$. Without loss of generality, assume that $y_2 = 1$. Let us show that y is adjacent to half of the vertices in a block

$$\left(\begin{array}{ccc|cc} * & \dots & * & a & b \\ 0 & \dots & 0 & c & d \end{array}\right).$$

We have

$$\begin{aligned} y + \left(\begin{array}{ccc|cc} * & \dots & * & a & b \\ 0 & \dots & 0 & c & d \end{array}\right) &= \left(\begin{array}{ccc|cc} * & \dots & * & a' & b' \\ 1 & \dots & y_{2e-4} & c' & d' \end{array}\right) \\ &= \left(\begin{array}{ccc|cc} 0 & \dots & * & a' & b' \\ 1 & \dots & y_{2e-4} & c' & d' \end{array}\right) \bigcup \left(\begin{array}{ccc|cc} 1 & \dots & * & a' & b' \\ 1 & \dots & y_{2e-4} & c' & d' \end{array}\right) \\ &= Y_0 \cup Y_1 \end{aligned}$$

Note that $|Y_0| = |Y_1|$, and the form Q has value 0 on one of the sets Y_0, Y_1 and value 1 on the other. We have proved that the switching preserves the number of common neighbours x and y, completing the proof of the theorem. $\square$

5.2.3 Construction 2

In the following subsections we will see that the smallest strictly Neumaier graph is the result of two consecutive switchings of the graph $VO^+(4, 2)$, but with different switching sets. We then generalise these new switchings to the graphs $VO^+(2e, 2)$ for larger e, and construct a second infinite sequence of strictly Neumaier graphs with the same edge-regular parameters as $VO^+(2e, 2)$.

The second construction of the smallest strictly Neumaier graph

Consider the graph $\Gamma_2 = VO^+(4, 2)$. Take the generator

$$W_1 = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix},$$

the vector

$$u = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and the coset

$$u + W_1 = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

Divide vertices of the 2-regular 4-cliques W_1 and $u + W_1$ into two parts as

$$W_1 = V_0 \cup V_1,$$

$$u + W_1 = V_2 \cup V_3,$$

where

$$V_0 = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} * & 1 \\ 0 & 0 \end{pmatrix},$$

$$V_2 = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} * & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that there are all possible edges between V_0 and V_2, there are all possible edges between V_1 and V_3, there are no edges between V_0 and V_3, and there are no edges between V_1 and V_2. Denote by Γ_2' the graph obtained from Γ_2 by switching edges between the cliques W_1 and $u + W_1$. Note that each of the sets $V_0 \cup V_3$ and $V_1 \cup V_2$ induces a 4-clique in Γ_2'.

The set

$$C := \begin{pmatrix} * & 0 \\ 1 & * \end{pmatrix}$$

induces a 2-regular 4-clique in the graph Γ_2' as well as in Γ_2 since the switching between W_1 and $u+W_1$ did not modify the neighbourhoods of the vertices from C. Moreover, $C \cap (W_1 \cup u + W_1) = \emptyset$ holds, and any vertex from C is adjacent to half of the vertices of each of the sets V_0, V_1, V_2, V_3. This means that the switching between the cliques $V_1 \cup V_2$, C and the switching between the cliques $V_0 \cup V_3$, C preserve the regularity of Γ_2'. Denote by Γ_2'' and Γ_2''' the graphs obtained from Γ_2' by applying these two switchings, respectively. One can prove that the graphs Γ_2'' and Γ_2''' are isomorphic to the smallest Neumaier graph. Now we show how can the adjacency matrix of the graph Γ_2'' be obtained from the adjacency matrix of Γ_2.

In this setting, the adjacency matrix of the affine polar graph $\Gamma_2 = VO^+(4, 2)$ can be written as in Table 5.5.

Switching edges between the cliques W_1, $u + W_1$ and then between the cliques $V_1 \cup V_2$, C is equivalent to inverting the highlighted entries in Table 5.5. This gives the strictly Neumaier graph $\Gamma_{2,2}$ on 16 vertices, whose adjacency matrix is presented in Table 5.6.

The notation in the rightmost column of Table 5.6 means the following. Two rows have the same letter if and only if they correspond to non-adjacent vertices having 8 common neighbours; two rows have the same number if and only if they correspond to non-adjacent vertices having 4 common neighbours. Otherwise, every two non-adjacent vertices have 6 common neighbours; every two adjacent vertices have 4 common neighbours.

The second generalisation of the smallest strictly Neumaier graph

In this subsection we generalise the construction above, presenting one more family of strictly Neumaier graphs.

For any $e \geq 2$, consider the affine polar graph $\Gamma_e = VO^+(2e, 2)$ and take the regular clique given by the generator

$$W_1 = \left(\begin{array}{ccc|cc} * & \dots & * & * & * \\ 0 & \dots & 0 & 0 & 0 \end{array}\right).$$

For the vector

$$u = \left(\begin{array}{ccc|cc} 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{array}\right),$$

	00 00	10 00	01 00	11 00	00 01	10 01	01 01	11 01	00 10	10 10	00 11	10 11	01 10	11 10	01 11	11 11
00 00	0	1	1	1	1	1	0	0	1	0	1	0	1	0	0	1
10 00	1	0	1	1	1	1	0	0	0	1	0	1	0	1	1	0
01 00	1	1	0	1	0	0	1	1	1	0	0	1	1	0	1	0
11 00	1	1	1	0	0	0	1	1	0	1	1	0	0	1	0	1
00 01	1	1	0	0	0	1	1	1	1	0	1	0	0	1	1	0
10 01	1	1	0	0	1	0	1	1	0	1	0	1	1	0	0	1
01 01	0	0	1	1	1	1	0	1	0	1	1	0	1	0	1	0
11 01	0	0	1	1	1	1	1	0	1	0	0	1	0	1	0	1
00 10	1	0	1	0	1	0	0	1	0	1	1	1	1	1	0	0
10 10	0	1	0	1	0	1	1	0	1	0	1	1	1	1	0	0
00 11	1	0	0	1	1	0	1	0	1	1	0	1	0	0	1	1
10 11	0	1	1	0	0	1	0	1	1	1	1	0	0	0	1	1
01 10	1	0	1	0	0	1	1	0	1	1	0	0	0	1	1	1
11 10	0	1	0	1	1	0	0	1	1	1	0	0	1	0	1	1
01 11	0	1	1	0	1	0	1	0	0	0	1	1	1	1	0	1
11 11	1	0	0	1	0	1	0	1	0	0	1	1	1	1	1	0

Table 5.5: The adjacency matrix, B_2, of Γ_2

take the regular clique

$$u + W_1 = \left(\begin{array}{ccc|cc} * & \dots & * & * & * \\ 0 & \dots & 0 & 0 & 1 \end{array} \right),$$

	00 00	10 00	01 00	11 00	00 01	10 01	01 01	11 01	00 10	10 10	00 11	10 11	01 10	11 10	01 11	11 11	
00 00	0	1	1	1	0	0	1	1	1	0	1	0	1	0	0	1	A1
10 00	1	0	1	1	0	0	1	1	0	1	0	1	0	1	1	0	B2
01 00	1	1	0	1	1	1	0	0	0	1	1	0	1	0	1	0	C3
11 00	1	1	1	0	1	1	0	0	1	0	0	1	0	1	0	1	D4
00 01	0	0	1	1	0	1	1	1	0	1	0	1	0	1	1	0	B1
10 01	0	0	1	1	1	0	1	1	1	0	1	0	1	0	0	1	A2
01 01	1	1	0	0	1	1	0	1	0	1	1	0	1	0	1	0	C4
11 01	1	1	0	0	1	1	1	0	1	0	0	1	0	1	0	1	D3
00 10	1	0	0	1	0	1	0	1	0	1	1	1	1	1	0	0	E5
10 10	0	1	1	0	1	0	1	0	1	0	1	1	1	1	0	0	F6
00 11	1	0	1	0	0	1	1	0	1	1	0	1	0	0	1	1	G7
10 11	0	1	0	1	1	0	0	1	1	1	1	0	0	0	1	1	H8
01 10	1	0	1	0	0	1	1	0	1	1	0	0	0	1	1	1	G8
11 10	0	1	0	1	1	0	0	1	1	1	0	0	1	0	1	1	H7
01 11	0	1	1	0	1	0	1	0	0	0	1	1	1	1	0	1	F5
11 11	1	0	0	1	0	1	0	1	0	0	1	1	1	1	1	0	E6

Table 5.6: The adjacency matrix, $B_{2,2}$, of $\Gamma_{2,2}$

which lies in the spread given by W_1. Divide W_1 and $u + W_1$ into two parts as

$$W_1 = V_0 \cup V_1,$$

$$u + W_1 = V_2 \cup V_3,$$

where

$$V_0 = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 0 & 0 \end{array}\right),$$

$$V_1 = \left(\begin{array}{ccc|cc} * & \dots & * & * & 1 \\ 0 & \dots & 0 & 0 & 0 \end{array}\right),$$

$$V_2 = \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 0 & 1 \end{array}\right),$$

$$V_3 = \left(\begin{array}{ccc|cc} * & \dots & * & * & 1 \\ 0 & \dots & 0 & 0 & 1 \end{array}\right).$$

Note that there are all possible edges between V_0 and V_2, there are all possible edges between V_1 and V_3, there are no edges between V_0 and V_3, and there are no edges between V_1 and V_2. Denote by Γ'_e the graph obtained from Γ_e by switching edges between the cliques W_1 and $u + W_1$. Note that each of the sets $V_0 \cup V_3$ and $V_1 \cup V_2$ induces a 2^e-clique in Γ'_e.

The set

$$C := \left(\begin{array}{ccc|cc} * & \dots & * & * & 0 \\ 0 & \dots & 0 & 1 & * \end{array}\right)$$

induces a 2^{e-1}-regular 2^e-clique in the graph Γ'_e as well as in Γ_e since the switching between W_1 and $u + W_1$ did not modify the neighbourhoods of the vertices from C. Moreover, $C \cap (W_1 \cup u + W_1) = \emptyset$ holds, and any vertex from C is adjacent to half of the vertices of each of the sets V_0, V_1, V_2, V_3. This means that the switching between the cliques $V_1 \cup V_2$, C and the switching between the cliques $V_0 \cup V_3$, C preserve the regularity of Γ'_e. Denote by $\Gamma_{e,2}$ the graph obtained from Γ'_e by switching edges between the cliques $W_1 \cup W_2$ and C. Let (v, k, λ, μ) be the parameters of the affine polar graph $\Gamma_e = VO^+(2e, 2)$ as a strongly regular graph.

Theorem 5.10. *The graph $\Gamma_{e,2}$ is a strictly Neumaier graph with parameters*

$$(2^{2e}, (2^{e-1} + 1)(2^e - 1), 2(2^{e-2} + 1)(2^{e-1} - 1); 2^{e-1}, 2^e).$$

Further, the number of common neighbours of two non-adjacent vertices in the graph *takes the values $\mu - 2^{e-1}, \mu$ and $\mu + 2^{e-1}$.*

Proof. For any $a, b, c, d \in \mathbb{F}_2$, let

$$\begin{matrix}\mathbf{ab}\\ \mathbf{cd}\end{matrix}$$

denote the set of matrices

$$\left(\begin{array}{ccc|cc} * & \dots & * & a & b \\ 0 & \dots & 0 & c & d \end{array}\right).$$

For the affine polar graph $\Gamma_e = VO^+(2e, 2)$, consider the subgraph Δ induced by the set of all matrices

$$\left(\begin{array}{ccc|cc} * & \dots & * & a & b \\ 0 & \dots & 0 & c & d \end{array}\right),$$

where a, b, c, d run over $\mathbb{F}_2$. The adjacency matrix of the subgraph Δ is presented by the block-matrix in Table 5.7, where K denotes the adjacency matrix of the complete graph on 2^{e-2} vertices; J denotes the all-ones matrix of size $2^{e-2} \times 2^{e-2}$; Z denotes the all-zeroes matrix of size $2^{e-2} \times 2^{e-2}$.

Switching edges between the cliques W_1, $u + W_1$ and then between the cliques $V_1 \cup V_2$, C is equivalent to inverting the highlighted entries in Table 5.7. This gives the submatrix of the adjacency matrix of $\Gamma_{e,2}$ presented in Table 5.8. Note that every switched edge connects vertices from the subgraph Δ. This means that the switching preserves all edges having a vertex outside of Δ.

Let (v, k, λ, μ) be the parameters of the affine polar graph $VO^+(2e, 2)$ as a strongly regular graph. We have to check that the obtained graph $\Gamma_{e,2}$ is a strictly Neumaier graph. Note that the vertices

$$\left(\begin{array}{ccc|cc} * & \dots & * & * & 1 \\ 0 & \dots & 0 & 1 & * \end{array}\right)$$

induce a 2^{e-1}-regular 2^e-clique in $\Gamma_{e,2}$ as well as in Γ_e. Let us check that any pair of vertices in $\Gamma_{e,2}$ is OK, i.e., any two adjacent vertices have λ common neighbours. Also, we investigate which values of μ occur in $\Gamma_{e,2}$.

Let us consider any two vertices inside of Δ. The notation in the right column of the matrix in Table 5.8 means the following. Two block-rows have the same letter if and only if any row from one block-row and any row from the other block-row

	00 00	10 00	01 00	11 00	00 01	10 01	01 01	11 01	00 10	10 10	00 11	10 11	01 10	11 10	01 11	11 11
00 00	K	J	J	J	J	J	Z	Z	J	Z	J	Z	J	Z	Z	J
10 00	J	K	J	J	J	J	Z	Z	Z	J	Z	J	Z	J	J	Z
01 00	J	J	K	J	Z	Z	J	J	J	Z	Z	J	J	Z	J	Z
11 00	J	J	J	K	Z	Z	J	J	Z	J	J	Z	Z	J	Z	J
00 01	J	J	Z	Z	K	J	J	J	J	Z	J	Z	Z	J	J	Z
10 01	J	J	Z	Z	J	K	J	J	Z	J	Z	J	J	Z	Z	J
01 01	Z	Z	J	J	J	J	K	J	Z	J	J	Z	J	Z	J	Z
11 01	Z	Z	J	J	J	J	J	K	J	Z	Z	J	Z	J	Z	J
00 10	J	Z	J	Z	J	Z	Z	J	K	J	J	J	J	J	Z	Z
10 10	Z	J	Z	J	Z	J	J	Z	J	K	J	J	J	J	Z	Z
00 11	J	Z	Z	J	J	Z	J	Z	J	J	K	J	Z	Z	J	J
10 11	Z	J	J	Z	Z	J	Z	J	J	J	J	K	Z	Z	J	J
01 10	J	Z	J	Z	Z	J	J	Z	J	J	Z	Z	K	J	J	J
11 10	Z	J	Z	J	J	Z	Z	J	J	J	Z	Z	J	K	J	J
01 11	Z	J	J	Z	J	Z	J	Z	Z	Z	J	J	J	J	K	J
11 11	J	Z	Z	J	Z	J	Z	J	Z	Z	J	J	J	J	J	K

Table 5.7: The adjacency matrix, B_e, of the subgraph Δ of Γ_e

correspond to non-adjacent vertices having $\mu + 2^{e-1}$ common neighbours; two block-rows have the same number if and only if any row from one block-row and any

	00 **00**	**10** **00**	**01** **00**	**11** **00**	**00** **01**	**10** **01**	**01** **01**	**11** **01**	**00** **10**	**10** **10**	**00** **11**	**10** **11**	**01** **10**	**11** **10**	**01** **11**	**11** **11**	
00 **00**	K	J	J	J	Z	Z	J	J	J	Z	J	Z	J	Z	Z	J	A1
10 **00**	J	K	J	J	Z	Z	J	J	Z	J	Z	J	Z	J	J	Z	B2
01 **00**	J	J	K	J	J	J	Z	Z	Z	J	J	Z	J	Z	J	Z	C3
11 **00**	J	J	J	K	J	J	Z	Z	J	Z	Z	J	Z	J	Z	J	D4
00 **01**	Z	Z	J	J	K	J	J	J	Z	J	Z	J	Z	J	J	Z	B1
10 **01**	Z	Z	J	J	J	K	J	J	J	Z	J	Z	J	Z	Z	J	A2
01 **01**	J	J	Z	Z	J	J	K	J	Z	J	J	Z	J	Z	J	Z	C4
11 **01**	J	J	Z	Z	J	J	J	K	J	Z	Z	J	Z	J	Z	J	D3
00 **10**	J	Z	Z	J	Z	J	Z	J	K	J	J	J	J	J	Z	Z	E5
10 **10**	Z	J	J	Z	J	Z	J	Z	J	K	J	J	J	J	Z	Z	F6
00 **11**	J	Z	J	Z	Z	J	J	Z	J	J	K	J	Z	Z	J	J	G7
10 **11**	Z	J	Z	J	J	Z	Z	J	J	J	J	K	Z	Z	J	J	H8
01 **10**	J	Z	J	Z	Z	J	J	Z	J	J	Z	Z	K	J	J	J	G8
11 **10**	Z	J	Z	J	J	Z	Z	J	J	J	Z	Z	J	K	J	J	H7
01 **11**	Z	J	J	Z	J	Z	J	Z	Z	Z	J	J	J	J	K	J	F5
11 **11**	J	Z	Z	J	Z	J	Z	J	Z	Z	J	J	J	J	J	K	E6

Table 5.8: The adjacency matrix, $B_{e,2}$, of the subgraph Δ of $\Gamma_{e,2}$

row from the other block-row correspond to non-adjacent vertices having $\mu - 2^{e-1}$ common neighbours. Otherwise, every two non-adjacent vertices corresponding to rows of this submatrix have μ common neighbours. Any two adjacent vertices have λ common neighbours. This means that all pairs of vertices inside of Δ are OK.

Let us consider any two vertices outside of Δ. Their neighbours and, consequently, their common neighbours are preserved by the switching. This means that all pairs of vertices outside of Δ are OK.

Let us consider a vertex x in Δ and a vertex y outside of Δ. If the neighbours of x are preserved by the switching, then x,y are OK. Assume that the neighbours of x are switched. Then the vertices x,y are OK since the vertex y is adjacent to half of the vertices of each block of Δ. In fact, the vertex y is given by a matrix

$$\left(\begin{array}{ccc|cc} y_1 & \cdots & y_{2e-5} & y_{2e-3} & y_{2e-1} \\ y_2 & \cdots & y_{2e-4} & y_{2e-2} & y_{2e} \end{array}\right),$$

where there is at least one non-zero among $y_2, y_4, \ldots, y_{2e-4}$. Without loss of generality, assume that $y_2 = 1$. Let us show that y is adjacent to half of the vertices in a block

$$\left(\begin{array}{ccc|cc} * & \ldots & * & a & b \\ 0 & \ldots & 0 & c & d \end{array}\right).$$

We have

$$\begin{aligned} y + \left(\begin{array}{ccc|cc} * & \ldots & * & a & b \\ 0 & \ldots & 0 & c & d \end{array}\right) &= \left(\begin{array}{ccc|cc} * & \ldots & * & a' & b' \\ 1 & \ldots & y_{2e-4} & c' & d' \end{array}\right) \\ &= \left(\begin{array}{ccc|cc} 0 & \ldots & * & a' & b' \\ 1 & \ldots & y_{2e-4} & c' & d' \end{array}\right) \bigcup \left(\begin{array}{ccc|cc} 1 & \ldots & * & a' & b' \\ 1 & \ldots & y_{2e-4} & c' & d' \end{array}\right) \\ &= Y_0 \cup Y_1. \end{aligned}$$

Note that $|Y_0| = |Y_1|$, and the form Q has value 0 on one of the sets Y_0, Y_1 and value 1 on the other. We have proved that the switching preserves the number of common neighbours x and y, completing the proof of the theorem. $\square$

Chapter 6

On equitable 2-partitions of the Johnson graph $J(n, 3)$

The relationship between association schemes and codes was the topic of the thesis of Delsarte [27]. Motivated by previous authors, Delsarte puts a particular emphasis on the Hamming and Johnson schemes. He makes a comment [27, pg 55], suggesting that there does not exist any non-trivial perfect codes in the Johnson graphs.

Martin [44] expands on the work of Delsarte by studying completely regular subsets in detail. In his work, Martin presents the relation between perfect codes and equitable 2-partitions more explicitly. In the literature, many substructures of regular graphs correspond to or are equivalent to equitable 2-partitions, for example, regular sets in Neumaier [50] (which we call (d, m)-regular sets) and perfect 2-colorings in Gavrilyuk Goryainov [33].

Equitable 2-partitions have been studied for several families of graphs, such as in the hypercubes by Fon-Der-Flaass [32] and in the generalized Petersen graphs by Alaeiyan and Karami [2]. Due to Delsarte's original work, there is continued interest in the Hamming graphs (e.g. Mogilnykh and Valyuzhenich [47]) and Johnson graphs (e.g. Mogilnykh [46] and Gavrilyuk and Goryainov [33]).

Any given equitable 2-partition of a graph is naturally associated with a non-principal eigenvalue of the graph. For each integer $k \geq 2$, the equitable 2-partitions of $J(n, k)$ associated to the second largest and smallest eigenvalues of $J(n, k)$ have been characterised (see Gavrilyuk and Goryainov [33] for references). For $k > 3$ and

$n \geq 2k$, all equitable 2-partitions of $J(n,k)$ associated to the third largest eigenvalue of $J(n,k)$ are characterised in an unpublished work of Vorob'ev [58]. The next open case is the equitable 2-partitions of the graphs $J(n,3)$ associated with the third largest eigenvalue of $J(n,3)$.

In this chapter we work on the classification of equitable 2-partitions of the Johnson graphs of diameter 3, $J(n,3)$. We note the known equitable 2-partitions and use block intersection numbers to prove there does not exist an equitable 2-partition for which one part is of diameter at most 2. Then we focus on the last eigenvalue for which the corresponding equitable 2-partitions have not been classified. In our approach, we analyse the local structure of such equitable 2-partitions by using an algebraic tool introduced by Garvrilyuk and Goryainov [33].

6.1 The Johnson graph $J(n,3)$

In this section we introduce the Johnson graphs $J(n,3)$ and note some properties of these graphs which we will use in our investigations.

For a positive integer p, we define $[p] := \{1,\ldots,p\}$. For positive integers p,q, the $p \times q$*-lattice* is the graph with vertex set $\{(i,j) : i \in [p], j \in [q]\}$, and two distinct vertices are joined by an edge precisely when they have the same value at one coordinate (note that the $n \times n$-lattice is the square lattice graph $L_2(n)$).

Let n be an integer, $n \geq 6$. The *Johnson graph* $J(n,3)$ has vertex set

$$\{K \subseteq [n] : |K| = 3\},$$

and distinct vertices K,L are adjacent if and only if $|K \cap L| = 2$. Throughout this chapter, the graph Γ will be the Johnson graph $J(n,3)$, where the value of n will be specified in advance. For any triple of distinct elements $a,b,c \in [n]$, let abc denote the set $\{a,b,c\}$. For distinct elements $i,j \in [n]$, denote by $ij*$ the set of subsets of $[n]$ of size 3 that contain both elements i and j. Note that $ij*$ induces a clique of size $n-2$ in $J(n,3)$.

The Johnson graph $J(n,3)$ is a distance-regular graph with diameter 3, so $J(n,3)$

is amply-regular. The amply-regular parameters of $J(n,3)$ are

$$\left(\binom{n}{3}, 3(n-3), n-2, 4\right),$$

and the eigenvalues of $J(n,3)$ are

$$\begin{aligned} k &= 3(n-3), \\ \theta_1 &= 2n-9, \\ \theta_2 &= n-7, \\ \theta_3 &= -3. \end{aligned}$$

For more information on Johnson graphs, see Brouwer, Cohen and Neumaier [14, Section 9.1].

It is known that the neighbourhood of any vertex in $J(n,3)$ is isomorphic to the $3 \times (n-3)$-lattice. In particular, there are three maximal cliques of size $n-3$ in the neighbourhood of abc, given by $ab*$, $ac*$ and $bc*$, and $n-3$ maximal cliques of size 3, given by the triples $\{abi, aci, bci\}$, where $i \in [n] \setminus \{a,b,c\}$. The *ab-row* of $\Gamma(abc)$ is the set $ab * \setminus abc$. For an element $i \in [n] \setminus \{a,b,c\}$, the *i-column* of $\Gamma(abc)$ is the set $\{abi, aci, bci\}$, and i is the *index* of this column.

6.2 Equitable 2-partitions

Let Δ be a graph, and $X = \{X_1, ..., X_q\}$ be a partition of the vertices of Δ. Then the sets X_i are called the *parts* of X, and X is called a *q-partition*. Let $A_{i,j}$ be the matrix $A(\Delta)$ restricted to the rows indexed by vertices in X_i, and columns indexed by vertices in X_j. Then there is an ordering of $V(\Delta)$ such that $A(\Delta)$ has the following block matrix form.

$$A(\Delta) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,q} \\ \vdots & \ddots & \vdots \\ A_{q,1} & \cdots & A_{q,q} \end{pmatrix}$$

Let $b_{i,j}$ be the average row-sum of $A_{i,j}$. The matrix

$$A(\Delta/X) = \begin{pmatrix} b_{1,1} & \cdots & b_{1,q} \\ \vdots & \ddots & \vdots \\ b_{q,1} & \cdots & b_{q,q} \end{pmatrix}$$

is called the *quotient matrix* of $A(\Delta)$ with respect to the partition X. The partition X is *equitable* if for each part X_i and every vertex $u \in X_i$ we have $|X_j \cap \Delta(u)| = b_{i,j}$ for every j.

We note that for a regular graph Δ, a (d, m)-regular set S gives the equitable 2-partition $\{S, V(\Delta) \setminus S\}$. Further, given an equitable 2-partition $\{S, V(\Delta) \setminus S\}$, the set S is a (d, m)-regular set, for some values of d and m.

Every equitable 2-partition in a regular graph can be naturally associated to an eigenvalue of the graph.

Lemma 6.1. *Let X be an equitable 2-partition of Δ with quotient matrix*

$$A(\Delta/X) = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$$

Then the eigenvalues of $A(\Delta/X)$ are given by $k = b_{1,1} + b_{1,2} = b_{2,1} + b_{2,2}$ and $\theta = b_{1,1} - b_{2,1} = b_{2,2} - b_{1,2}$, where $\theta \in Spec(\Delta), \theta \neq k$.

Proof. This is a simple application of eigenvalue interlacing of quotient matrices and a routine calculation of the eigenvalues of a 2x2 matrix. For details, see Gavrilyuk and Goryainov [33]. $\square$

Let Δ be a k-regular graph and θ be a real number, $\theta \neq k$. An equitable 2-partition X of Δ is *θ-equitable* if $A(\Delta/X)$ has eigenvalue θ . By Lemma 6.1, the quotient matrix of an equitable 2-partition X has eigenvalues k and θ, where θ is an eigenvalue of Δ. Therefore, we can enumerate equitable partitions of a regular graph by enumerating θ-equitable 2-partitions for each eigenvalue $\theta \in \mathrm{Spec}(\Gamma)$.

6.3 Known equitable 2-partitions in $J(n, 3)$

The θ-equitable 2-partitions of $J(n, 3)$ have been classified for the eigenvalues $\theta = \theta_1$ and θ_3. For more references and information on these partitions, see Gavrilyuk and

Goryainov [33]. The open case corresponds to the eigenvalue θ_2.

Analysis of the classification problem for small values of n can be found in Mogilnykh [46] and Avgustinovich and Mogilynkh [4, 5]. For example, they find all of the possible quotient matrices for an equitable 2-partition in $J(8, 3)$. Avgustinovich and Mogilynkh give many constructions, but the author is not aware of any classification results of θ_2-equitable 2-partitions in $J(n, 3)$ for any $n > 6$.

In Avgustinovoich and Mogilnykh [5], an equitable 3-partition of $J(2m, 3)$ was constructed for all $m \geq 3$. This construction was used to produce three families of θ_2-equitable 2-partitions of $J(2m, 3)$. Here we give a detailed presentation of this construction.

Let $U = \{u_1, \ldots, u_m\}$ and $W = \{w_1, \ldots, w_m\}$ be sets of integers such that $U \cup W = [2m]$ (i.e. U and W partition the set $[2m]$). Let $\Delta_{\{U,W\}}$ be the graph with vertices $U \cup W$ and edge set $\{u_i w_j : i \neq j\}$. In other words, $\Delta_{\{U,W\}}$ is constructed by taking the complete bipartite graph with parts U, W, and then removing the edges $u_1 w_1, u_2 w_2, \ldots, u_m w_m$.

There are three "types" of unordered triples of vertices in $\Delta_{\{U,W\}}$. Any set of three distinct vertices $abc \subseteq [2m]$ lies in one of the following sets:

$$
\begin{aligned}
X_1 &= \{u_i u_j u_k : i, j, k \text{ distinct}\} \cup \{w_i w_j w_k : i, j, k \text{ distinct}\}.\\
X_2 &= \{u_i u_j w_i : i, j \text{ distinct}\} \cup \{w_i w_j u_i : i, j \text{ distinct}\}.\\
X_3 &= \{u_i u_j w_k : i, j, k \text{ distinct}\} \cup \{w_i w_j u_k : i, j, k \text{ distinct}\}.
\end{aligned}
$$

Now we regard the triples of vertices of $\Delta_{\{U,W\}}$ as the vertices of $J(2m, 3)$. The partition $\Pi = \{X_1, X_2, X_3\}$ gives a 3-partition of the vertex set of $J(2m, 3)$.

Consider $\Gamma := J(2m, 3)$ and vertices $t_1 = u_1 u_2 u_3, t_2 = u_1 u_2 w_1, t_3 = u_1 u_2 w_3$, so $t_i \in X_i$ for each i. The following $3 \times (2m-3)$-arrays represent the $3 \times (2m-3)$-grids induced by $\Gamma(t_1)$, $\Gamma(t_2)$ and $\Gamma(t_3)$. The indexes of columns are given below the braces. The rows are given by the comments on the right. The entries are from the set $\{1, 2, 3\}$, and an entry equals to j whenever the corresponding vertex belongs to the part X_j.

$$\Gamma(t_1):\quad \begin{array}{ccccc}
1\ \ldots\ 1 & 3\ \ldots\ 3 & 2 & 2 & 3 \\
1\ \ldots\ 1 & 3\ \ldots\ 3 & 2 & 3 & 2 \\
\underbrace{1\ \ldots\ 1}_{U\backslash\{u_1,u_2,u_3\}} & \underbrace{3\ \ldots\ 3}_{W\backslash\{w_1,w_2,w_3\}} & \underbrace{3}_{w_1} & \underbrace{2}_{w_2} & \underbrace{2}_{w_3}
\end{array}
\begin{array}{l}
\leftarrow u_1u_2\text{-row} \\
\leftarrow u_1u_3\text{-row} \\
\leftarrow u_2u_3\text{-row} \\

\end{array}$$

$$\Gamma(t_2):\quad \begin{array}{ccc}
1\ \ldots\ 1 & 3\ \ldots\ 3 & 2 \\
2\ \ldots\ 2 & 2\ \ldots\ 2 & 2 \\
\underbrace{3\ \ldots\ 3}_{U\backslash\{u_1,u_2\}} & \underbrace{3\ \ldots\ 3}_{W\backslash\{w_1,w_2\}} & \underbrace{2}_{w_2}
\end{array}
\begin{array}{l}
\leftarrow u_1u_2\text{-row} \\
\leftarrow u_1w_1\text{-row} \\
\leftarrow u_2w_1\text{-row} \\

\end{array}$$

$$\Gamma(t_3):\quad \begin{array}{ccccc}
1\ \ldots\ 1 & 3\ \ldots\ 3 & 1 & 2 & 2 \\
3\ \ldots\ 3 & 3\ \ldots\ 3 & 2 & 2 & 3 \\
\underbrace{3\ \ldots\ 3}_{U\backslash\{u_1,u_2,u_3\}} & \underbrace{3\ \ldots\ 3}_{W\backslash\{w_1,w_2,w_3\}} & \underbrace{2}_{u_3} & \underbrace{3}_{w_1} & \underbrace{2}_{w_2}
\end{array}
\begin{array}{l}
\leftarrow u_1u_2\text{-row} \\
\leftarrow u_1w_3\text{-row} \\
\leftarrow u_2w_3\text{-row} \\

\end{array}$$

With the knowledge of these neighbourhoods and the symmetry of the graph $\Delta_{\{U,W\}}$, we deduce that $\Pi = \{X_1, X_2, X_3\}$ is an equitable 3-partition of $J(2m, 3)$.

Lemma 6.2. *Let* $\Pi = \{X_1, X_2, X_3\}$ *be the partition of the vertices of* $\Gamma = J(2m, 3)$ *defined above. Then* Π *is equitable, and has quotient matrix*

$$A(\Gamma/\Pi) = \begin{pmatrix} 3m-9 & 6 & 3m-6 \\ m-2 & 2m-1 & 3m-6 \\ m-2 & 6 & 5m-13 \end{pmatrix}.$$

Proof. Any vertex $abc \in \Gamma$ lies in X_j for exactly one value $j \in \{1, 2, 3\}$. Then there is a permutation of $[2m]$ such that the $3 \times (2m-3)$-array of $\Gamma(abc)$ is the array of t_j in the above. This observation shows that the partition Π is equitable, and we can use the arrays of the neighbourhoods to determine the quotient matrix. $\square$

Now we construct three different equitable 2-partitions by merging the parts of the partition Π. Let $\Pi_1 := \{X_2 \cup X_3, X_1\}$, $\Pi_2 := \{X_1 \cup X_3, X_2\}$ and $\Pi_3 = \{X_3, X_1 \cup X_2\}$.

Lemma 6.3. *Let Π_j be the partitions of the vertices of $\Gamma = J(2m, 3)$ defined above. Then the partitions Π_j are equitable, and have quotient matrices*

$$A(\Gamma/\Pi_1) = \begin{pmatrix} 5m-7 & m-2 \\ 3m & 3m-9 \end{pmatrix},$$

$$A(\Gamma/\Pi_2) = \begin{pmatrix} 6m-15 & 6 \\ 4m-8 & 2m-1 \end{pmatrix},$$

$$A(\Gamma/\Pi_3) = \begin{pmatrix} 5m-13 & m+4 \\ 3m-6 & 3m-3 \end{pmatrix}.$$

Proof. This follows from Avgustinovich and Mogilnykh [5, Lemma 1], which shows that because the quotient matrix $A(\Gamma/\Pi)$ has equal non-diagonal entries for each column, we can merge parts of the equitable 3-partition to get equitable 2-partitions. The quotient matrices also follow easily from this argument. □

6.4 Diameter 2 (d, m)-regular sets

As noted in Section 6.2, a (d, m)-regular set of a regular graph Δ gives an equitable 2-partition. In this section, we will prove that for $n > 6$ and any d, m, there does not exist a (d, m)-regular set S of $\Gamma = J(n, 3)$ such that $\Gamma[S]$ has diameter at most 2. This implies there is no equitable 2-partitions of $J(n, 3)$ for $n > 6$, such that either part of the partition induces a subgraph of diameter at most 2. To prove this result we will use the block intersection numbers found in Chapter 2.

Let Δ be an amply regular graph with parameters (v, k, λ, μ). Consider a set $S \subset V(\Delta)$, $s := |S|$ and assume $\{S, V(\Delta) \setminus S\}$ is a θ-equitable 2-partition. Then there exists d, m such that S is a (d, m)-regular set. By Lemma 6.1, we know that $m = d - \theta > 0$ and $s = vm/(k - \theta)$.

We partition the pairs of distinct vertices of S into the following sets;

$$\begin{aligned} E(S) &:= \{\{u, w\} \subseteq S : u, w \text{ at distance 1 in } \Delta\}. \\ F(S) &:= \{\{u, w\} \subseteq S : u, w \text{ at distance 2 in } \Delta\}. \\ G(S) &:= \{\{u, w\} \subseteq S : u, w \text{ at distance at least 3 in } \Delta\}. \end{aligned}$$

Let $e(S) := |E(S)|, f(S) := |F(S)|, g(S) := |G(S)|$ be the size of these sets.

From definition we see that λ_0 is the number of vertices in $V(\Delta) \setminus S$, so

$$\lambda_0 = v - s$$

We can also see that for any $u \in S$, we have $\lambda_{\{u\}} = k - d$, so

$$\lambda_1 = k - d$$

Now consider $\binom{s}{2}\lambda_2$. Note that this is the number of paths of length 2 with distinct end points in S, and midpoint in $V(\Delta) \setminus S$. Consider any pair of vertices $u, w \in S$.

1. If $\{u, w\} \in E(S)$ (adjacent in Δ), there are exactly λ walks of length 2 between them in the graph Δ.

2. If $\{u, w\} \in F(S)$ (at distance 2 in Δ), there are exactly μ walks of length 2 between them in the graph Δ.

3. If $\{u, w\} \in G(S)$ (at distance at least 3), there are no walks of length 2 between them.

Therefore, in the graph Δ there are a total of $e(S)\lambda + f(S)\mu$ walks of length 2 between distinct vertices in S. We can also count the number walks of length 2 between distinct vertices in S where the midpoint also lies in S. This is exactly

$$\sum_{u \in S} \binom{|\Delta(u)|}{2} = s\binom{d}{2}.$$

We deduce from this that

$$\binom{s}{2}\lambda_2 = e(S)\lambda + f(S)\mu - s\binom{d}{2}.$$

We also know that

$$\begin{aligned} e(S) &= \frac{sd}{2} \\ e(S) + f(S) + g(S) &= \binom{s}{2}, \end{aligned}$$

so we can eliminate $e(S), f(S)$ to get

$$2\binom{s}{2}\lambda_2 = (\lambda - \mu + 1)sd + s(s-1)\mu - sd^2 - 2g(S)\mu \tag{6.1}$$

Lemma 6.4. *Let S be a (d, m)-regular set of Δ, where $m = d - \theta$. Then*

$$\frac{2g(S)\mu(k-\theta)^2}{vm} = (v\mu - (k-\theta)(k+\theta-\lambda+\mu))m - (k-\theta)(\theta^2 - (\lambda-\mu)\theta + \mu - k) \tag{6.2}$$

Proof. We continue to manipulate the equation for λ_2. First, we use equation (6.1), and then we use equation (2.14) and substitute the values for λ_1 and $m = d - \theta$.

$$\begin{aligned}
\frac{2g(S)\mu}{s} &= -\lambda_2(s-1) + (\lambda - \mu + 1)d + (s-1)\mu - d^2 \\
&= -(m-1)\lambda_1 + (\lambda - \mu + 1)d + (s-1)\mu - d^2 \\
&= (k-d)(\theta+1) - kd + (\lambda - \mu + 1)d + (s-1)\mu
\end{aligned}$$

Now we use the identity $s = vm/(k-\theta)$ and multiply both sides by $k - \theta$ to get

$$\begin{aligned}
\frac{2g(S)\mu(k-\theta)^2}{vm} &= (k-\theta)(k-d)(\theta+1) - (k-\theta)kd \\
&+ (k-\theta)(\lambda-\mu+1)d + \mu(v(d-\theta) - (k-\theta)) \\
&= (v\mu - (k-\theta)(k+\theta-\lambda+\mu))d \\
&+ (k-\theta)(k(\theta+1) - \mu) - v\mu\theta \\
&= (v\mu - (k-\theta)(k+\theta-\lambda+\mu))m \\
&- (k-\theta)(\theta^2 - (\lambda-\mu)\theta + \mu - k).
\end{aligned}$$

□

Now we concentrate on the case of the Johnson graph $\Gamma = J(n, 3)$ and use Equation (6.2) to investigate θ_j-equitable 2-partitions for $j = 1, 2, 3$. First we restrict our interest to θ_1-equitable 2-partitions.

Lemma 6.5. *Let S be a (d,m)-regular set of Γ, where $m = d - \theta_1$. Then*

$$(n-4)(n-5)(2m/3-1)m = \frac{48g(S)}{(n-1)(n-2)}.$$

Proof. Beginning with Equation (6.2), and substituting for the parameters of the Johnson graph $J(n,3)$ and the value of θ_1, we see that

$$\begin{aligned}\frac{48n^2g(S)}{n(n-1)(n-2)m} &= \frac{2n}{3}(n^2-9n+20)m - n(n^2-9n+20)\\ &= n(n-4)(n-5)(2m/3-1).\end{aligned}$$

□

Now we consider the case where we have a θ_2-equitable 2-partition.

Lemma 6.6. *Let S be a (d,m)-regular set of Γ, where $m = d - \theta_2$. Then*

$$(n-5)(n-6)(m+\frac{12}{n-6})m = \frac{288g(S)}{n(n-2)m}.$$

Proof. Substituting the parameters of the Johnson graph $J(n,3)$ into the above, and dividing both sides by $2(n-1)/3$, we see that

$$\begin{aligned}\frac{288g(S)}{n(n-2)m} &= (n^2-11n+30)m + 12(n-5)\\ &= (n-5)(n-6)m + 12(n-5).\end{aligned}$$

For $n > 6$ we deduce that

$$\frac{288g(S)}{n(n-2)} = (n-5)(n-6)(m+\frac{12}{n-6})m.$$

□

Finally we consider the θ_3-equitable partitions.

Lemma 6.7. *Suppose Γ contains a (d,m)-regular set, where $m = d - \theta_3$. Then*

$$((n^2-10n+27)m-18)m = \frac{648g(S)}{n(n-1)}.$$

Proof. This follows similarly to Lemma 6.6. □

In particular, if $g(S) = 0$ we see that there are no possible positive values for m when $n > 6$. This covers the case where $\Gamma[S]$ has diameter at most 2.

Theorem 6.8. *Let $\Gamma = J(n,3)$, where $n > 6$. Suppose $X = \{X_1, X_2\}$ is an equitable 2-partition. Then $\Gamma[X_1]$ is disconnected, or has diameter strictly greater than 2.*

Proof. We know that X_1 is a (d,m)-regular set for some d, m and $m = d - \theta_j$ for some $j \in \{1,2,3\}$. Suppose $\Gamma[X_1]$ is connected with diameter at most 2. By definition, we have $g(X_1) = 0$.

Consider the cases $j = 1$ and 2, and the corresponding equations from Lemmas 6.5 and 6.6. By solving for integer values of m in each case, we see that $m = 0$, contradicting the fact that $m > 0$.

Finally, consider the case when $j = 3$, and the equation in Lemma 6.7 with $g(X_1) = 0$. As $n > 6$ and $m > 0$, we see that the possible cases for n and m are $n = 7$ and $m = 1$, or $n = 9$ and $m = 3$. We also have $d = m + \theta_3$ is non-negative, which forces $m = 3$ and $d = 0$. As $\Gamma[X_1]$ is connected, X_1 must be a single vertex. But a single vertex does not give a $(0,3)$-regular set in $J(9,3)$, so we have a contradiction. □

6.5 Local structure of equitable 2-partitions

Gavrilyuk and Goryainov [33] use a an algebraic tool to analyse the structure of a equitable 2-partition in $J(n,3)$. In particular, they classify all equitable 2-partitions in in $J(n,3)$ when n is odd, and all equitable 2-partitions in $J(n,3)$ for which both parts have the same size (such a partition corresponds to a symmetric quotient matrix). In this section, we introduce the tools used by Garvrilyuk and Goryainov [33], and use them to analyse the remaining cases of θ_2-equitable 2-partitions of $J(n,3)$.

Let θ be a non-principal eigenvalue of $\Gamma = J(n,3)$ and let $X = \{X_1, X_2\}$ be a θ-equitable 2-partition of $J(n,3)$. Let the quotient matrix of X be

$$A(\Gamma/X) = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.$$

For a vertex $u \in V(\Gamma)$, the X_1-*indicator function* on u is

$$\overline{u} := \begin{cases} 1 & \text{if } u \in X_1; \\ 0 & \text{if } u \in X_2, \end{cases}$$

and for a set of vertices $U \subseteq V(\Gamma)$ we define

$$\overline{U} := \sum_{u \in U} \overline{u}$$

We have the following identities, relating the indicator of a vertex abc of Γ and the number of vertices in each row of $\Gamma(abc)$ that are in X_1.

Lemma 6.9. *For a vertex abc of $J(n,3)$, the following equality holds*

$$\overline{abc} \cdot (\theta + 3) + p_{21} = \overline{ab*} + \overline{ac*} + \overline{bc*}. \tag{6.3}$$

Proof. This can be shown by using a simple counting argument and the relationship between the quotient matrix and the number $\overline{\Gamma(abc)}$. For the full details, see Gavrilyuk and Goryainov [33, Equation (2)]. □

Lemma 6.10. *For any four distinct elements $a, b, c, d \in [n]$, the following condition holds:*

$$\overline{ab*} - \overline{cd*} = \frac{\theta + 3}{2}(\overline{abc} + \overline{abd} - \overline{acd} - \overline{bcd}). \tag{6.4}$$

Proof. This result is effectively a rearrangement and application of Gavrilyuk and Goryainov [33, Lemma 3.5].

By Lemma 6.9, we have the following equalities:

$$\overline{abc} \cdot (\theta + 3) + p_{21} = \overline{ab*} + \overline{ac*} + \overline{bc*},$$

$$\overline{abd} \cdot (\theta + 3) + p_{21} = \overline{ab*} + \overline{ad*} + \overline{bd*},$$

$$-\overline{acd} \cdot (\theta + 3) - p_{21} = -\overline{ac*} - \overline{ad*} - \overline{cd*},$$

$$-\overline{bcd} \cdot (\theta + 3) - p_{21} = -\overline{bc*} - \overline{bd*} - \overline{cd*}.$$

Then sum up these four equalities and divide by 2 to see the result. □

Lemma 6.10 shows that for any two vertex-disjoint maximum cliques $ab*$ and $cd*$ in $J(n,3)$, the difference $\overline{ab*} - \overline{cd*}$ is determined by the eigenvalue θ and the four values $\overline{abc}$, $\overline{abd}$, $\overline{acd}$ and $\overline{bcd}$.

Lemma 6.11. *For any five distinct elements $a, b, c, d, e \in [n]$, the following condition holds:*

$$\overline{ab*} - \overline{ac*} = \frac{\theta+3}{2}(\overline{abd} + \overline{abe} + \overline{cde} - \overline{acd} - \overline{ace} - \overline{bde}). \tag{6.5}$$

Proof. By Lemma 6.10, we have the following equalities:

$$\overline{ab*} - \overline{cd*} = \frac{\theta+3}{2}(\overline{abc} + \overline{abd} - \overline{acd} - \overline{bcd}),$$

$$\overline{cd*} - \overline{be*} = \frac{\theta+3}{2}(\overline{bcd} + \overline{cde} - \overline{bce} - \overline{bde}),$$

$$\overline{be*} - \overline{ac*} = \frac{\theta+3}{2}(\overline{abe} + \overline{bce} - \overline{abc} - \overline{ace}).$$

Then we sum up these three equalities to see the result. □

6.5.1 Local structure of θ_2-equitable 2-partitions

Let us now consider the case when the partition X is θ_2-equitable. We will assume that $p_{11} \geq p_{22}$. By Lemma 6.1, we have

$$4(n-4) = k + \theta_2 = p_{11} + p_{22},$$

so we must have $p_{11} \geq 2n-8$. If $p_{11} = 2n-8$, we see that $p_{11} = p_{22}$ and $p_{12} = p_{21} = n-1$. All θ_2-equitable 2-partitions with such a quotient matrix are found in Gavrilyuk and Goryainov [33]. For the rest of the chapter we assume $p_{11} \geq 2n-7$.

Let abc be a vertex of Γ, such that $abc \in X_1$. By Equation (6.5), we have

$$\overline{ab*} - \overline{ac*} = \frac{n-4}{2}(\overline{abd} + \overline{abe} + \overline{cde} - \overline{acd} - \overline{ace} - \overline{bde}). \tag{6.6}$$

Using this and the fact that $0 \leq \overline{ij*} \leq n-2$, we see that the difference $\overline{ab*} - \overline{ac*}$ is equal to $h(n-4)/2$, where $h \in \{-2,-1,0,1,2\}$.

Now we fix a vertex $abc \in V(\Gamma)$. Without loss of generality, we may assume that $\overline{ab*} \geq \overline{ac*} \geq \overline{bc*}$. Now consider the possible tuples, $(\overline{ab*} - \overline{ac*}, \overline{ac*} - \overline{bc*})$. We have the following possibilities:

(I) $(n-4, 0)$.

(II) $\left(\frac{n-4}{2}, \frac{n-4}{2}\right)$.

(III) $(0, n-4)$.

(IV) $\left(\frac{n-4}{2}, 0\right)$

(V) $\left(0, \frac{n-4}{2}\right)$.

(VI) $(0, 0)$.

We will investigate the cases given above and find that some cannot occur for large enough values of n.

6.5.2 Case analysis

To understand the arguments in this section, we will be using arrays of the following form. Consider the vertex abc of Γ and let $X = \{X_1, X_2\}$ be a θ_2-equitable partition of Γ. The *nb-array* of abc is a $3 \times (2m-3)$-array corresponding to the neighbourhood $\Gamma(abc)$, and at each vertex $u \in \Gamma(abc)$ the corresponding entry of the array is equal to $\overline{u}$. The rows are given by the comments on the right. If needed, the indices of columns are given below the braces.

$$\Gamma(abc): \begin{array}{ccccccccccccl} \overline{abd} & \overline{abe} & \overline{abf} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \leftarrow ab\text{-row} \\ \overline{acd} & \overline{ace} & \overline{acf} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \leftarrow ac\text{-row} \\ \underbrace{\overline{bcd}}_{d} & \underbrace{\overline{bce} \quad \overline{bcf}}_{\{e,f\}} & & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \leftarrow bc\text{-row} \end{array}$$

When considering the cases (I)-(VI), we can use Equation (6.6) to deduce restrictions on the corresponding nb-arrays. Note that we have assumed that $\overline{ab*} \geq \overline{ac*} \geq \overline{bc*}$.

Lemma 6.12. *Let abc be a vertex of Γ and $\overline{abc} = 1$ Then:*

1. If $\overline{ab*} = \overline{ac*}$, there are no indices $d, e \in [n] \setminus \{a, b, c\}$ such that $\overline{abd} = \overline{abe} = 1$ and $\overline{acd} = \overline{ace} = 0$.

2. If $\overline{ab*} > \overline{ac*}$, there are no indices $d, e \in [n] \setminus \{a, b, c\}$ such that

(a) $\overline{acd} = 1$ and $\overline{abe} = \overline{abd} = \overline{ace} = 0$,

(b) $\overline{abd} = 0$ and $\overline{abe} = \overline{acd} = \overline{ace} = 1$, or

(c) $\overline{acd} = \overline{ace} = 1$ and $\overline{abd} = \overline{abe} = 0$.

Proof. 1. Suppose $d, e \in [n] \setminus \{a, b, c\}$ such that $\overline{abd} = \overline{abe} = 1$ and $\overline{acd} = \overline{ace} = 0$. Then by Equation (6.6),

$$0 = 2 + \overline{cde} - \overline{bde} > 0,$$

giving a contradiction.

2.(a) Suppose $d, e \in [n] \setminus \{a, b, c\}$ such that $\overline{acd} = 1$ and $\overline{abe} = \overline{abd} = \overline{ace} = 0$. By Equation (6.6),

$$0 < \overline{cde} - 1 - \overline{bde} < 1,$$

giving a contradiction. The result of 2.(b) and (c) follows similarly. □

Now we use Lemma 6.12 to enumerate all possible nb-arrays in each case. In fact, Cases (I) and (II) are shown to be impossible for all $n > 12$. In Cases (III) and (IV) we can characterise the quotient matrix in terms of n.

Case (I)

In this case, we observe that $p_{11} = \overline{\Gamma(abc)}$ is at most $(n-3) + 1 + 1 = n - 1$. Therefore $n - 1 \geq 2n - 7$ by assumption, and so $n \leq 6$.

Case (II)

In this case, we observe that $p_{11} = \overline{\Gamma(abc)}$ is at most $(n-3)+(n-2)/2+1 = 3(n-2)/2$. Therefore $3(n-2)/2 \geq 2n - 7$ by assumption, and so $n \leq 8$.

Case (III)

In this case, we observe that $p_{11} = \overline{\Gamma(abc)}$ is at most $(n-3)+n-3+1 = 2n-5$. Furthermore, we must have $\overline{ab*} \geq n-2$, as otherwise $p_{11} \leq (n-4)+(n-4)+0 = 2n-8$. Therefore, $\overline{ab*} = n-2$ and $p_{11} = 2n-5$.

From this we deduce that the only possible nb-arrays (up to reordering) are the following.

$$\Gamma(abc) : \begin{array}{ccccccccccccl} 1 & 1 & 1 & \dots & \dots & \dots & \dots & \dots & 1 & 1 & 1 & 1 & \leftarrow ab\text{-row} \\ 1 & 1 & 1 & \dots & \dots & \dots & \dots & \dots & 1 & 1 & 1 & 1 & \leftarrow ac\text{-row} \\ 1 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 & 0 & \leftarrow bc\text{-row} \end{array}$$

Case (IV)

In this case, we observe that $p_{11} = \overline{\Gamma(abc)}$ is at most $(n-3)+(n-2)/2+(n-2)/2 = 2n-5$. Furthermore, we must have $\overline{ab*} \geq n-2$, as otherwise $p_{11} \leq (n-3-1) + 2(n-3-1-(n-4)/2) = 2n-8$. Therefore, $\overline{ab*} = n-2$ and $p_{11} = 2n-5$.

Using Lemma 6.12 1., we see that any two distinct columns of the nb-array of $\Gamma(abc)$ (up to reordering) cannot look like one of the following pairs.

$$\begin{array}{cc} 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{array} \quad \text{or} \quad \begin{array}{ccl} 1 & 1 & \leftarrow ab\text{-row} \\ 1 & 1 & \leftarrow ac\text{-row} \\ 0 & 0 & \leftarrow bc\text{-row} \end{array}$$

Furthermore, if we have a single column from one of the above pairs present in the nb-array, then a column of the other pair must also be present, as $\overline{ac*} = \overline{bc*}$.

Therefore, the only possible nb-arrays (up to reordering) are the following.

(i)

$$\Gamma(abc) : \begin{array}{cccccccccl} 1 & \dots & \dots & 1 & 1 & 1 & 1 & \dots & \dots & 1 \ \leftarrow ab\text{-row} \\ 1 & \dots & \dots & 1 & 1 & 0 & 0 & \dots & \dots & 0 \ \leftarrow ac\text{-row} \\ 1 & \dots & \dots & 1 & 1 & 0 & 0 & \dots & \dots & 0 \ \leftarrow bc\text{-row} \end{array}$$

(ii)

$$\Gamma(abc): \begin{array}{ccccccccccl} 1 & 1 & 1 & \dots & \dots & 1 & 1 & \dots & \dots & 1 & \leftarrow ab\text{-row} \\ 1 & 0 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ac\text{-row} \\ 0 & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow bc\text{-row} \end{array}$$

Case (V)

Let $t_V = \overline{ab*} - 1$. Then $p_{11} = \overline{\Gamma(abc)} = 3t_V - (n-4)/2$ and so $t_V \geq (5n-18)/6$.

Using Lemma 6.12, we see that any two distinct columns of the nb-array of $\Gamma(abc)$ (up to reordering) cannot look like the following pairs.

$$\underbrace{\begin{array}{ccccc} 1 & 1 & & 0 & 0 \\ 0 & 0 & \text{or} & 1 & 1 \\ * & * & & * & * \end{array}}_{\text{by Lemma 6.12 1.}} \qquad \underbrace{\begin{array}{ccccc} 0 & 0 & & * & * \\ * & * & \text{or} & 0 & 0 \\ 1 & 0 & & 1 & 0 \end{array}}_{\text{by Lemma 6.12 2(a).}}$$

$$\underbrace{\begin{array}{ccccc} 0 & 1 & & * & * \\ * & * & \text{or} & 0 & 1 \\ 1 & 1 & & 1 & 1 \end{array}}_{\text{by Lemma 6.12 2(b).}} \qquad \underbrace{\begin{array}{ccccc} * & * & & 0 & 0 \\ 0 & 0 & \text{or} & * & * \\ 1 & 1 & & 1 & 1 \end{array}}_{\text{by Lemma 6.12 2(c).}}$$

(where any entry $*$ can take either of the values 0 or 1).

Now suppose we have a column

$$\begin{array}{cl} * & \leftarrow ab\text{-row} \\ 0 & \leftarrow ac\text{-row} \\ 1 & \leftarrow bc\text{-row} \end{array}$$

By the above restrictions on pairs of columns, any other column of the nb-array must look like

$$\begin{array}{cl} * & \leftarrow ab\text{-row} \\ 1 & \leftarrow ac\text{-row} \\ 0 & \leftarrow bc\text{-row} \end{array}$$

This shows us that $\overline{ac*} = n-3$ and $\overline{bc*} = 2$. But this gives a contradiction to the fact that $\overline{ac*} - \overline{bc*} = (n-4)/2$ and $n > 6$.

By a similar argument, we can show that we cannot have the column

$$\begin{array}{ll} 0 & \leftarrow ab\text{-row} \\ * & \leftarrow ac\text{-row} \\ 1 & \leftarrow bc\text{-row} \end{array},$$

and so the possible columns of the nb-array are

$$\begin{array}{cccccl} 1 & 1 & 0 & 1 & 0 & \leftarrow ab\text{-row} \\ 1 & 1 & 0 & 0 & 1 & \leftarrow ac\text{-row} \\ 1 & 0 & 0 & 0 & 0 & \leftarrow bc\text{-row} \end{array},$$

By the restrictions on pairs of columns from Lemma 6.12 1., the last two columns in the list can occur at most once. If one of the last two columns are present in the nb-array, the other must be present, as $\overline{ab*} = \overline{ac*}$ (this will be called case (ii)).

From the discussion above, we deduce that the only possible nb-arrays (up to reordering) are the following.

(i)

$$\Gamma(abc) : \begin{array}{cccccccccccl} 1 & \dots & \dots & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ab\text{-row} \\ 1 & \dots & \dots & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ac\text{-row} \\ 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 & \leftarrow bc\text{-row} \end{array}$$

(This case occurs in the partition Π_1 found in Lemma 6.3, for which $p_{11} = 5n/2 - 7$.)

(ii)

$$\Gamma(abc) : \begin{array}{cccccccccccccc} 1 & 0 & 1 & \dots & \dots & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ab\text{-row} \\ 0 & 1 & 1 & \dots & \dots & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ac\text{-row} \\ 0 & 0 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 & \leftarrow bc\text{-row} \end{array}$$

(This case occurs in the partition Π_3 found in Lemma 6.3, for which $p_{11} = 5n/2 - 13$.)

Case (VI)

Let $t_{VI} = \overline{ab*} - 1$. Then $p_{11} = \overline{\Gamma(abc)} = 3t_{VI}$ and so $t_{VI} \geq (2n-7)/3$.

Using Lemma 6.12 1., we see that any two distinct columns of the nb-array of $\Gamma(abc)$ (up to reordering) cannot look like one of the following pairs.

$$\begin{array}{cc} 1 & 1 \\ 0 & 0 \\ * & * \end{array} \qquad \begin{array}{cc} 0 & 0 \\ 1 & 1 \\ * & * \end{array} \qquad \begin{array}{cc} 1 & 1 \\ * & * \\ 0 & 0 \end{array} \qquad \begin{array}{cc} 0 & 0 \\ * & * \\ 1 & 1 \end{array} \qquad \begin{array}{cc} * & * \\ 1 & 1 \\ 0 & 0 \end{array} \qquad \begin{array}{cc} * & * \\ 0 & 0 \\ 1 & 1 \end{array}$$

(where any entry $*$ can take either of the values 0 or 1).

Suppose we have a column in the nb-array with exactly two entries equal to 1 (this will split into the cases (ii) and (iv) in the following). Without loss of generality, let this column be as follows.

$$\begin{array}{cl} 1 & \leftarrow ab\text{-row} \\ 1 & \leftarrow ac\text{-row} \\ 0 & \leftarrow bc\text{-row} \end{array}$$

As $\overline{ac*} = \overline{bc*}$, at least one other column must be of the form

$$\begin{array}{cl} * & \leftarrow ab\text{-row} \\ 0 & \leftarrow ac\text{-row} \\ 1 & \leftarrow bc\text{-row} \end{array}$$

Suppose we are in the case (this will be called case (iv)) that we have columns

$$\begin{array}{ccl} 1 & 1 & \leftarrow ab\text{-row} \\ 1 & 0 & \leftarrow ac\text{-row} \\ 0 & 1 & \leftarrow bc\text{-row} \end{array}$$

We use the fact that $\overline{ab*} = \overline{ac*}$ and the restrictions on the columns to find that we must have the three columns

$$\begin{array}{cccl} 1 & 1 & 0 & \leftarrow ab\text{-row} \\ 1 & 0 & 1 & \leftarrow ac\text{-row} \;. \\ 0 & 1 & 1 & \leftarrow bc\text{-row} \end{array}$$

From here, it is straightforward to see that each of the remaining columns must all three entries equal.

Now suppose we are in the case (this will called case (ii)) where instead, we have columns

$$\begin{array}{ccl} 1 & 0 & \leftarrow ab\text{-row} \\ 1 & 0 & \leftarrow ac\text{-row} \\ 0 & 1 & \leftarrow bc\text{-row} \end{array}$$

Using the restrictions on pairs of columns, we see that any other column cannot be of the form

$$\begin{array}{cccl} * & & 0 & \leftarrow ab\text{-row} \\ 0 & \text{or} & * & \leftarrow ac\text{-row} \\ 1 & & 1 & \leftarrow bc\text{-row} \end{array}$$

This means that any non-zero entry which contributes positively to the sum $\overline{bc*}$ must be in a column with all entries equal to 1. As $\overline{ab*} = \overline{ac*} = \overline{bc*}$, we deduce that all columns with at least one entry equal to 1 must have all entries equal to 1.

Now suppose we have no columns with exactly two entries equal to 1 (this will split into the cases (i) and (iii) in the following). Further suppose there is a column with exactly one entry equal to 1 (this will be called case (iii)). Without loss of generality, let this column be as follows.

$$\begin{array}{cl} 1 & \leftarrow ab\text{-row} \\ 0 & \leftarrow ac\text{-row} \\ 0 & \leftarrow bc\text{-row} \end{array}$$

By the restrictions on the pairs of columns, the assumption we have no columns with two entries equal to 1, and $\overline{ab*} = \overline{ac*} = \overline{bc*}$, we must have the three columns.

$$\begin{array}{cccl} 1 & 0 & 0 & \leftarrow ab\text{-row} \\ 0 & 1 & 0 & \leftarrow ac\text{-row} \\ 0 & 0 & 1 & \leftarrow bc\text{-row} \end{array}$$

From here, it is straightforward to see that each of the remaining columns must all three entries equal.

The last case is where all entries of a single column are equal (this will be called case (i)).

In the discussion above, we use Lemma 6.12 1. to deduce that the only possible nb-arrays (up to reordering) are the following.

(i)

$$\Gamma(abc): \begin{array}{cccccccccccl} 1 & 1 & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ab\text{-row} \\ 1 & 1 & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ac\text{-row} \\ 1 & 1 & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow bc\text{-row} \end{array}$$

(ii)

$$\Gamma(abc): \begin{array}{cccccccccccl} 1 & 0 & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ab\text{-row} \\ 1 & 0 & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ac\text{-row} \\ 0 & 1 & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow bc\text{-row} \end{array}$$

(iii)

$$\Gamma(abc): \begin{array}{cccccccccccl} 1 & 0 & 0 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ab\text{-row} \\ 0 & 1 & 0 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ac\text{-row} \\ 0 & 0 & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow bc\text{-row} \end{array}$$

(This case occurs in the partition Π_2 found in Lemma 6.3, for which $p_{11} = 3(n-5)$.)

(iv)

$$\Gamma(abc): \begin{array}{cccccccccccl} 1 & 1 & 0 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ab\text{-row} \\ 1 & 0 & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow ac\text{-row} \\ 0 & 1 & 1 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow bc\text{-row} \end{array}$$

6.5.3 Removing Cases (III) and (IV)(ii)

From now on, we will assume we do not see the cases (I) or (II) for any vertex in Γ. Note that when $n > 8$, we know that these cases cannot occur. We will then prove that the cases (III) and (IV)(ii) cannot occur. First we will prove that these cases always occur together.

Lemma 6.13. *Let abc be a vertex of Γ with $\overline{abc} = 1$, and let $d, e \in [n] \setminus \{a, b, c\}$ be* *distinct. Then we have nb-array*

$$\Gamma(abc): \begin{array}{ccccccccccccc} 1 & 1 & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & 1 & 1 & 1 & 1 & \leftarrow ab\text{-}row \\ 1 & 1 & 1 & \ldots & \ldots & \ldots & \ldots & \ldots & 1 & 1 & 1 & 1 & \leftarrow ac\text{-}row \\ \underbrace{1}_{d} & 0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 & 0 & 0 & 0 & \leftarrow bc\text{-}row \end{array}$$

if and only if we have nb-array

$$\Gamma(abe): \begin{array}{ccccccccccc} 1 & 1 & 1 & \ldots & \ldots & 1 & 1 & \ldots & \ldots & 1 & \leftarrow ab\text{-}row \\ 1 & 0 & 1 & \ldots & \ldots & 1 & 0 & \ldots & \ldots & 0 & \leftarrow ae\text{-}row \\ \underbrace{0}_{c} & \underbrace{1}_{d} & \underbrace{1 \quad \ldots \quad \ldots \quad 1} & & & & \underbrace{0 \quad \ldots \quad \ldots \quad 0} & & & & \leftarrow be\text{-}row \end{array}$$

Proof. ($\Longrightarrow$) Suppose we have the nb-array for abc. Then by applying Equation (6.6) to the rows $ab*, bc*$, we have

$$2 = \overline{abd} + \overline{abe} + \overline{cde} - \overline{bcd} - \overline{bce} - \overline{ade} = 1 + \overline{cde} - \overline{ade}.$$

Therefore, we must have $\overline{cde} = 1, \overline{ade} = 0$. Applying Equation (6.6) to the rows $ab*, ac*$, we have $\overline{bde} = 1$.

Further application of Equation (6.6) to the rows $ab*, ae*$, and using the above, we see that

$$\overline{ab*} - \overline{ae*} = \frac{n-4}{2}(\overline{abc} + \overline{abe} + \overline{cde} - \overline{ace} - \overline{ade} - \overline{bcd}) = \frac{n-4}{2},$$

and so $\overline{ae*} = n/2$.

Now consider the nb-array of abe. We know the values for columns with indices c, d. We have assumed case (II) does not occur, so the nb-array of abe must be in case (IV). Using our knowledge of the c-column and d-column, we see the nb-array of abe is in Case (IV)(ii).

($\Longleftarrow$) Suppose the nb-array of abe is of the form above.

Applying Equation (6.6) to the rows $ab*, ad*$ and $ac*, ab*$, we see that

$$\overline{ab*} - \overline{ad*} = \frac{n-4}{2}(\overline{abc} + \overline{abe} + \overline{cde} - \overline{acd} - \overline{ade} - \overline{bce}) = \frac{n-4}{2}(2 + \overline{cde} - \overline{acd}),$$

and

$$\overline{ac*} - \overline{ab*} = \frac{n-4}{2}(\overline{acd} + \overline{ace} + \overline{bde} - \overline{abd} - \overline{abe} - \overline{cde}) = \frac{n-4}{2}(\overline{acd} - \overline{cde}).$$

Summing these two together, we see that

$$\overline{ac*} - \overline{ad*} = n - 4$$

and thus $\overline{ac*} \geq n - 4$.

Consider the nb-array of vertex abc. As the difference $\overline{ab*} - \overline{ac*}$ is an integer multiple of $(n-4)/2$ and $\overline{ab*} = n-2$, we must have $\overline{ac*} = n-2$, and the nb-array of abc must be in Case (III). $\square$

Now we will work to prove that Case (III) leads to a contradiction, proving that cases (III) and (IV)(ii) cannot occur (when n is large enough).

Lemma 6.14. *Let abc be a vertex of Γ with $\overline{abc} = 1$, and let $d \in [n] \setminus \{a, b, c\}$ be such that we have nb-array*

$$\Gamma(abc) : \begin{array}{cccccccccccccc} 1 & 1 & 1 & \dots & \dots & \dots & \dots & \dots & 1 & 1 & 1 & 1 & \leftarrow \textit{ab-row} \\ 1 & 1 & 1 & \dots & \dots & \dots & \dots & \dots & 1 & 1 & 1 & 1 & \leftarrow \textit{ac-row} \\ \underbrace{1}_{d} & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 & 0 & \leftarrow \textit{bc-row} \end{array}$$

Then:

1. *The nb-array of abd is*

$$\Gamma(abd) : \begin{array}{cccccccccccccc} 1 & 1 & 1 & \dots & \dots & \dots & \dots & \dots & 1 & 1 & 1 & 1 & \leftarrow \textit{ab-row} \\ 1 & 1 & 1 & \dots & \dots & \dots & \dots & \dots & 1 & 1 & 1 & 1 & \leftarrow \textit{bd-row} \\ \underbrace{1}_{c} & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 & 0 & \leftarrow \textit{ad-row} \end{array} .$$

2. *For any $i, j \in [n] \setminus \{a, b, c, d\}$ distinct, we have $\overline{aij} = \overline{bij} = \overline{cij}$.*

3. *For any $e \in [n] \setminus \{a, b, c, d\}$, there exists $I_\omega, I_\beta \subseteq [n]$ such that we have the*

following nb-arrays:

$$\Gamma(abe): \begin{array}{ccccccccccc} 1 & 1 & 1 & \dots & \dots & 1 & 1 & \dots & \dots & 1 & \leftarrow \textit{ab-row} \\ 1 & 0 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow \textit{ae-row} \\ \underbrace{0}_{c} & \underbrace{1}_{d} & \multicolumn{4}{c}{\underbrace{1 \quad \dots \quad \dots \quad 1}_{I_\omega}} & \multicolumn{4}{c}{\underbrace{0 \quad \dots \quad \dots \quad 0}_{I_\beta}} & \leftarrow \textit{be-row} \end{array}$$

$$\Gamma(ace): \begin{array}{ccccccccccc} 1 & 1 & 1 & \dots & \dots & 1 & 1 & \dots & \dots & 1 & \leftarrow \textit{ac-row} \\ 1 & 0 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 & \leftarrow \textit{ae-row} \\ \underbrace{0}_{b} & \underbrace{1}_{d} & \multicolumn{4}{c}{\underbrace{1 \quad \dots \quad \dots \quad 1}_{I_\omega}} & \multicolumn{4}{c}{\underbrace{0 \quad \dots \quad \dots \quad 0}_{I_\beta}} & \leftarrow \textit{ce-row} \end{array}$$

Proof. 1. In Lemma 6.13, we prove that $\overline{ac*} - \overline{ad*} = n-4$ and $\overline{ac*} = n-2$, so $\overline{ad*} = 2$. As $p_{11} = 2n-5$ and $\overline{ab*} = n-2$, we deduce that $\overline{bd*} = n-2$. We have completely determined the nb-array of abd.

2. Let $i, j \in [n] \setminus \{a,b,c,d\}$. Applying Equation (6.6) to the rows $ab*, bc*$, we have

$$2 = \overline{abi} + \overline{abj} + \overline{cij} - \overline{bci} - \overline{bcj} - \overline{aij} = 2 + \overline{cij} - \overline{aij}.$$

Therefore, we have $\overline{cij} = \overline{aij}$. Similarly, applying Equation (6.6) to the rows $ab*, ac*$, we deduce $\overline{cij} = \overline{bij}$.

3. This follows from 1. and 2., and our previous knowledge of the nb-arrays of abc, abe.

□

We show that $|I_\omega| < 3$, which means that $n \leq 8$.

Lemma 6.15. *Let abc be a vertex of Γ with $\overline{abc} = 1$, and let $d \in [n] \setminus \{a,b,c\}$ be such that we have nb-array*

$$\Gamma(abc): \begin{array}{cccccccccccccc} 1 & 1 & 1 & \dots & \dots & \dots & \dots & \dots & 1 & 1 & 1 & 1 & \leftarrow \textit{ab-row} \\ 1 & 1 & 1 & \dots & \dots & \dots & \dots & \dots & 1 & 1 & 1 & 1 & \leftarrow \textit{ac-row} \\ \underbrace{1}_{d} & 0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 & 0 & \leftarrow \textit{bc-row} \end{array}$$

Then $n \leq 8$.

Proof. We take d, e, I_ω, I_β as in Lemma 6.14. Suppose $|I_\omega| \geq 3$, and let $i, j, k \in I_\omega$ be distinct.

First we find the nb-array of bde. By assumption, we know $\overline{abc}, \overline{abe}, \overline{bcd}, \overline{bce}$, and by Lemma 6.14 1., we know $\overline{ace}, \overline{ade}$. By Lemma 6.14 1., we also know the bd-row, and by Lemma 6.14 3., we have determined the be-row. We note that the nb-array of bde must then be in Case (IV)(ii), and we have the nb-array

$$\Gamma(bde): \begin{array}{ccccccccc} 1 & 1 & 1 & \dots & \dots & 1 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & \dots & \dots & 1 & 0 & \dots & \dots & 0 \\ \underbrace{0}_{a} & \underbrace{1}_{c} & \underbrace{1 & \dots & \dots & 1}_{I_\omega} & \underbrace{0 & \dots & \dots & 0}_{I_\beta} \end{array} \begin{array}{l} \leftarrow bd\text{-row} \\ \leftarrow be\text{-row} \\ \leftarrow de\text{-row} \end{array} \tag{6.7}$$

Now consider bei. By nb-array (6.7), we know $be*$. Applying Equation (6.6),

$$\begin{aligned} \overline{ei*} - \overline{bi*} &= \frac{n-4}{2}(\overline{aei} + \overline{cei} + \overline{abc} - \overline{abi} - \overline{bci} - \overline{ace}) \\ &= \frac{n-4}{2}(1 + 1 + 1 - 1 - 0 - 1) \\ &= \frac{n-4}{2} \end{aligned}$$

(here we use $\overline{bei} = \overline{cei}$ by Lemma 6.14 3.). Therefore $\overline{ei*} = n - 2$ and we have the nb-array

$$\Gamma(bei): \begin{array}{cccccccccc} 1 & 1 & 1 & 1 & \dots & \dots & 1 & 1 & \dots & \dots & 1 \\ 1 & 0 & 1 & * & \dots & \dots & * & * & \dots & \dots & * \\ \underbrace{1}_{a} & \underbrace{0}_{c} & \underbrace{1}_{d} & \underbrace{1 & \dots & \dots & 1}_{I_\omega \backslash \{i\}} & \underbrace{0 & \dots & \dots & 0}_{I_\beta} \end{array} \begin{array}{l} \leftarrow ei\text{-row} \\ \leftarrow bi\text{-row} \\ \leftarrow be\text{-row} \end{array} \tag{6.8}$$

Now we find the values of $\overline{bij}, \overline{bik}$. By nb-array (6.8), we have

$$\overline{be*} - \overline{bi*} = \frac{n-4}{2}(\overline{abe} + \overline{bce} + \overline{aci} - \overline{abi} - \overline{bci} - \overline{ace}) = 0$$

Therefore

$$\frac{n-4}{2} = \overline{ei*} - \overline{bi*}$$

$$
\begin{aligned}
&= \frac{n-4}{2}(\overline{aei} + \overline{eij} + \overline{abj} - \overline{abi} - \overline{bij} - \overline{aej}) \\
&= \frac{n-4}{2}(1 - \overline{bij}).
\end{aligned}
$$

Therefore $\overline{bij} = 0$ and $\overline{bik} = 0$, and we have nb-array

$$
\Gamma(bei): \begin{array}{cccccccccccc l}
1 & 1 & 1 & 1 & 1 & 1 & \dots & \dots & 1 & 1 & \dots & \dots & 1 & \leftarrow ei\text{-row} \\
1 & 0 & 1 & 0 & 0 & 0 & \dots & \dots & 0 & * & \dots & \dots & * & \leftarrow bi\text{-row} \\
\underbrace{1}_{a} & \underbrace{0}_{c} & \underbrace{1}_{d} & \underbrace{1}_{j} & \underbrace{1}_{k} & \multicolumn{4}{c}{\underbrace{1 \quad \dots \quad \dots \quad 1}_{I_\omega \backslash \{i,j,k\}}} & \multicolumn{4}{c}{\underbrace{0 \quad \dots \quad \dots \quad 0}_{I_\beta}} & \leftarrow be\text{-row}
\end{array}
$$

Applying equation (6.6) to the rows $bi*, be*$, we have

$$
\overline{be*} - \overline{bi*} = \frac{n-4}{2}(2 + \overline{ijk} - \overline{ejk}),
$$

which is non-zero. This implies that the nb-array bei is in Case (II), a contradiction. Therefore $|I_\omega| \leq 2$. By looking at the nb-array of abe we see that $|I_\omega| = (n-4)/2$, and so $n \leq 8$. □

In particular, if $n > 8$ we have proven that the cases (III) and (IV)(ii) do not occur. We have not yet used this approach to investigate the cases (IV)(i) and (VI)(i),(iii) and (iv), which have not yet been observed in a construction of a θ_2-equitable 2-partition in $J(n,3)$.

Going forward, we will attempt to reduce the number of possible cases by using the tools we use above, as well as other algebraic tools. For example, orthogonality of eigenfunctions is used by Vorob'ev [58] to classify θ_2-equitable 2-partitions in $J(n,k)$, where $k > 3$, and certain minimality properties are used by Bailey et al. [6] to classify θ-equitable partitions in Latin square graphs, where θ is the largest non-principal eigenvalue of the graphs. The author finds it conceivable that applying these techniques will show that the only θ_2-equitable 2-partitions in $J(n,3)$ are the partitions described in Section 6.3.

Chapter 7

Conclusions

We have studied three problems involving regular induced subgraphs of edge-regular graphs. In each problem, we assume additional structure on the graphs or subgraphs we investigate. Each of these problems leads to individual areas of interest.

In Chapter 3, we determine upper and lower bounds on the order of a d-regular induced subgraph of any strongly regular graph with parameters (v, k, λ, μ), and find that our new bounds are at least as good as the bounds on the order of a d-regular induced subgraph of a k-regular graph determined by Haemers [40]. We also compute that our bounds improve on Haemers' bounds for many strongly regular graphs on at most 1300 vertices. The arguments found in the latter part of Chapter 3 give good reason to believe our bounds on regular induced subgraphs of strongly regular graphs can improve on Haemers' bounds infinitely often.

In Chapter 4 we introduce terminology and present fundamental results on the subject of Neumaier graphs. Using results which force a Neumaier graph to be strongly regular, we determine the smallest strictly Neumaier graph, which is vertex-transitive and has order 16. This graph provides answers to questions A and B asked by Greaves and Koolen [37].

In Chapter 5 we generalise a construction of Neumaier graphs found by Greaves and Koolen [36], and present two new infinite sequences of strictly Neumaier graphs. These sequences each have first element the unique smallest strictly Neumaier graph, and both show that the nexus of a clique in a strictly Neumaier graph is not bounded above by some constant number. Furthermore, these sequences of graphs are closely related to some well-known strongly regular graphs.

In Chapter 6, we collect results on the known equitable 2-partitions of the graphs $J(n,3)$. Then we prove for $n > 6$, there does not exist an equitable 2-partition for which one part induces a subgraph of diameter at most 2. Concentrating on the case of the last eigenvalue for which the corresponding equitable 2-partitions have not been classified, we analyse the local structure of an assumed equitable 2-partition. In this process we consider several cases, and prove the non-existence of a partition with certain local structure.

After observing several constructions of strictly Neuamier graphs, interest in the possible algebraic and combinatorial properties of Neumaier graphs has emerged. One of the major open problems in this area is to determine the existence of a strictly Neumaier graph containing an m-regular clique, for every integer $m > 0$. For this, it may be possible to use a computational approach to develop a better understanding of the problem, and give us some small examples of graphs with the properties we are looking for.

The topic of edge-regular graphs with regular induced subgraphs encompasses many problems found in the literature. For example, many graphs have been characterised as the only graphs having extremal parameters or certain regular induced subgraphs. The bounds and parameter conditions presented throughout my thesis offer further opportunities to investigate their corresponding extremal cases.

There is also potential for both computational and theoretical tools to be developed in the area of generation and enumeration of regular graphs with given additional structure. For example, we could study graphs with a specific group of automorphisms or graphs containing a given regular subgraph or equitable 2-partition. Some well-known strongly regular graphs are interesting in this regard, as they have a sporadic simple group as a group of automorphisms. Such algorithms have been developed for strongly regular graphs, where the graphs have known spectral properties. The software GAP would be a great instrument to explore these problems further and to generalise these tools to larger classes of graphs.

Some well-known results in analytic number theory and computational experiments suggest that our bounds on regular induced subgraphs of strongly regular graphs can improve on Haemers' bounds infinitely often. I plan to prove our bounds improve on Haemers' bounds infinitely often, and possibly that the bounds can be better than Haemers' bounds by an arbitrary amount.

The problem of classifying all equitable 2-partitions of the Johnson graphs $J(n, 3)$ remains open. We plan to continue our analysis of the local structure of the equitable 2-partitions of $J(n, 3)$ and use other eigenvalue techniques (e.g. orthogonality conditions) to finish the classification. This will involve determining the existence of partitions with structure corresponding to the open cases (III), (IV)(i), (VI)(i), (VI)(ii) and (VI)(iv) found in Section 6.5.1. We can also use the tools we apply to the Johnson graphs to investigate equitable 2-partitions of other families of distance-regular graphs.

Algebraic graph theory continues to be an active field of research, with many interesting open problems still to be tackled. The objects we investigate in this research area are often highly regular or symmetric. Such objects give an abundance of interesting theoretical results, and present an opportunity for efficient computational exploration. This motivates me to continue studying the intriguing open problems in the area, with the ambition to develop useful results and tools of both theoretical and computational flavours.

Appendix A

AGT: A GAP package

A.1 Background

This is a manual for the AGT package version 0.1 [30].

The AGT package contains methods used for the determination of various algebraic and regularity properties of graphs, as well as certain substructures of graphs. The package also contains a library of strongly regular graphs, intended to be a useful resource for computational experiments.

All of the functions in this package deal with finite simple graphs in GRAPE format [55]. Behind the scenes, we also use the Digraphs package [25] to efficiently store and access the graphs in the strongly regular graph library.

A.1.1 Licence

The AGT package is free software; you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation.

A.1.2 Package dependencies

The AGT package requires the following GAP packages:

- GAPDoc [43], version 1.6 or higher;
- DESIGN [56], version 1.7 or higher;
- GRAPE [55], version 4.8 or higher;
- Digraphs [25], version 0.12.2 or higher.

Each of the above packages are part of the standard GAP distribution.

A.1.3 Installation

The AGT package is part of the standard GAP distribution as of version 4.11.0. To install the AGT package manually, you will need to download the most recent `tar.gz` file, found at `https://gap-packages.github.io/agt/`. Once downloaded, you can install the package by following the instructions found in the GAP reference manual, chapter 76.

A.1.4 Initialisation

Once correctly installed, you can load the AGT package at the GAP prompt by typing the following.

```
gap> LoadPackage("agt");
true
```

A.2 Regular graphs

In this section we give functions used to identify graphs with various regularity properties and determine their parameters.

A.2.1: RGParameters

▶ RGParameters(*gamma*) *(function)*

Returns: A list or `fail`.

Given a graph *gamma*, this function returns the regular graph parameters of *gamma*. If *gamma* is not a regular graph, the function returns `fail`.

e.g.

```
gap> gamma:=EdgeOrbitsGraph(Group((2,3,4,5)),[[1,2],[2,1]]);;
gap> RGParameters(gamma);
fail
gap> gamma:=HammingGraph(3,4);;
gap> RGParameters(gamma);
[ 64, 9 ]
```

A.2.2: IsRG

▶ IsRG(*gamma*) *(function)*

Returns: `true` or `false`.

Given a graph *gamma*, this function returns `true` if *gamma* is a regular graph, and `false` otherwise.

e.g.

```
gap> gamma:=NullGraph(Group(()),5);;
gap> IsRG(gamma);
true
gap> gamma:=EdgeOrbitsGraph(Group((2,3,4,5)),[[1,2],[2,1]]);;
gap> IsRG(gamma);
false
gap> gamma:=TriangularGraph(6);;
gap> IsRG(gamma);
true
```

A.2.3: IsFeasibleRGParameters

▸ `IsFeasibleRGParameters([v,k])` *(function)*

Returns: `true` or `false`.

Given a list of integers of length 2, *`[v,k]`*, this function returns `true` if (*`v`*, *`k`*) is a feasible parameter tuple for a regular graph. Otherwise, the function returns `false`.

The tuple (v, k) is a *feasible* parameter tuple for a regular graph if it satisfies the following well-known conditions:

- $v > k \geq 0$;
- 2 divides vk.

Any regular graph must have parameters that satisfy these conditions (see Brouwer, Cohen and Neumaier [14]).

e.g.

```
gap> IsFeasibleRGParameters([15,9]);
false
gap> IsFeasibleRGParameters([16,9]);
true
```

A.2.4: ERGParameters

▸ `ERGParameters(gamma)` *(function)*

Returns: A list or `fail`.

Given a graph *`gamma`*, this function returns the edge-regular graph parameters of *`gamma`*. If *`gamma`* is not an edge-regular graph, the function returns `fail`.

e.g.

```
gap> gamma:=NullGraph(Group(()),5);;
gap> ERGParameters(gamma);
fail
gap> gamma:=JohnsonGraph(7,3);;
gap> ERGParameters(gamma);
[ 35, 12, 5 ]
```

A.2.5: IsERG

▶ `IsERG(gamma)` (function)

Returns: `true` or `false`.

Given a graph *gamma*, this function returns `true` if *gamma* is an edge-regular graph, and `false` otherwise.

e.g.

```
gap> gamma:=NullGraph(Group(()),5);;
gap> IsERG(gamma);
false
gap> gamma:=JohnsonGraph(7,3);;
gap> IsERG(gamma);
true
```

A.2.6: IsFeasibleERGParameters

▶ `IsFeasibleERGParameters([v,k,a])` (function)

Returns: `true` or `false`.

Given a list of integers of length 3, *[v,k,a]*, this function returns `true` if (*v*, *k*, *a*) is a feasible parameter tuple for an edge-regular graph. Otherwise, the function returns `false`.

The tuple (v, k, a) is a *feasible* parameter tuple for an edge-regular graph if it satisfies the following well-known conditions:

- (v, k) is a feasible regular graph parameter tuple;

- $k > a \geq 0$;
- 2 divides ka and 6 divides vka;
- $v - 2k + a \geq 0$.

Any edge-regular graph must have parameters which satisfy these conditions (see Brouwer, Cohen and Neumaier [14]).

e.g.

```
gap> IsFeasibleERGParameters([15,9,6]);
false
gap> IsFeasibleERGParameters([16,9,4]);
true
```

A.2.7: SRGParameters

▸ SRGParameters(*gamma*) (function)

Returns: A list or `fail`.

Given a graph *gamma*, this function returns the strongly regular graph parameters of *gamma*. If *gamma* is not a strongly regular graph, the function returns `fail`.

e.g.

```
gap> gamma:=CompleteGraph(Group(()),5);;
gap> SRGParameters(gamma);
fail
gap> gamma:=JohnsonGraph(5,3);;
gap> SRGParameters(gamma);
[ 10, 6, 3, 4 ]
```

A.2.8: IsSRG

▸ IsSRG(*gamma*) (function)

Returns: `true` or `false`.

Given a graph *gamma*, this function returns `true` if *gamma* is a strongly regular graph, and `false` otherwise.

e.g.

```
gap> gamma:=CompleteGraph(Group(()),5);;
gap> IsSRG(gamma);
false
gap> gamma:=JohnsonGraph(5,3);;
gap> IsSRG(gamma);
true
```

A.2.9: IsFeasibleSRGParameters

▸ `IsFeasibleSRGParameters([v,k,a,b])` *(function)*

Returns: `true` or `false`.

Given a list of integers of length 4, `[v,k,a,b]`, this function returns `true` if (`v, k, a, b`) is a feasible parameter tuple for a strongly regular graph. Otherwise, this function returns `false`.

The tuple (v, k, a, b) is a *feasible* parameter tuple for a strongly regular graph if it satisfies the following well-known conditions:

- (v, k, a) is a feasible edge-regular graph parameter tuple;
- $k \geq b$;
- $(v - k - 1)b = k(k - a - 1)$;
- $v - 2 - 2k + b \geq 0$;
- the formulae for the multiplicities of the eigenvalues of a strongly regular graph with these parameters evaluate to positive integers (see Brouwer and Haemers [15]).

Any strongly regular graph must have parameters which satisfy these conditions (see Brouwer, Cohen and Neumaier [14]).

e.g.

```
gap> IsFeasibleSRGParameters([15,9,4,7]);
false
gap> IsFeasibleSRGParameters([10,3,0,1]);
true
```

A.3 Spectra of graphs

In this section we give methods for investigating the eigenvalues of a graph.

A.3.1: LeastEigenvalueInterval

▶ `LeastEigenvalueInterval(gamma, eps)` (operation)
▶ `LeastEigenvalueInterval(parms. eps)` (operation)

Returns: A list.

Given a graph *gamma* and rational number *eps*, this function returns an interval containing the least eigenvalue of *gamma*.

Given feasible strongly regular graph parameters *parms* and rational number *eps*, this function returns an interval containing the least eigenvalue of a strongly regular graph with parameters *parms*.

The interval returned is in the form of a list, *[y,z]* of rationals $y \leq z$ with the property that $z - y \leq eps$. If the eigenvalue is rational this function will return a list *[y,z]*, where $y = z$.

e.g.

```
gap> gamma:=EdgeOrbitsGraph(Group((1,2,3,4,5)),[[1,2],[2,1]]);;
gap> LeastEigenvalueInterval(gamma,1/10);
[ -13/8, -25/16 ]
gap> parms:=SRGParameters(gamma);
[ 5, 2, 0, 1 ]
gap> LeastEigenvalueInterval(parms,1/10);
[ -13/8, -25/16 ]
gap> LeastEigenvalueInterval(JohnsonGraph(7,3),1/20);
[ -3, -3 ]
```

A.3.2: SecondEigenvalueInterval

▶ `SecondEigenvalueInterval(`*gamma, eps*`)` *(operation)*
▶ `SecondEigenvalueInterval(`*parms. eps*`)` *(operation)*

Returns: A list.

Given a regular graph ***gamma*** and rational number ***eps***, this function returns an interval containing the second largest eigenvalue of ***gamma***.

Given feasible strongly regular graph parameters ***parms*** and rational number ***eps***, this function returns an interval containing the second largest eigenvalue of a strongly regular graph with parameters ***parms***.

The interval returned is in the form of a list, *[y,z]* of rationals $y \leq z$ with the property that $z - y \leq eps$. If the eigenvalue is rational this function will return a list *[y,z]*, where $y = z$.

e.g.

```
gap> gamma:=EdgeOrbitsGraph(Group((1,2,3,4,5)),[[1,2],[2,1]]);;
gap> SecondEigenvalueInterval(gamma,1/10);
[ 9/16, 5/8 ]
gap> parms:=SRGParameters(gamma);
[ 5, 2, 0, 1 ]
gap> SecondEigenvalueInterval(parms,1/10);
[ 9/16, 5/8 ]
gap> SecondEigenvalueInterval(JohnsonGraph(7,3),1/20);
[ 5, 5 ]
```

A.3.3: LeastEigenvalueFromSRGParameters

▶ `LeastEigenvalueFromSRGParameters([v,k,a,b])` *(function)*

Returns: An integer or an element of a cyclotomic field.
Given feasible strongly regular graph parameters `[v, k, a, b]` this function returns the least eigenvalue of a strongly regular graph with parameters (`v,k,a,b`). If the eigenvalue is integer, the object returned is an integer. If the eigenvalue is irrational, the object returned lies in a cyclotomic field.

e.g.

```
gap> LeastEigenvalueFromSRGParameters([5,2,0,1]);
E(5)^2+E(5)^3
gap> LeastEigenvalueFromSRGParameters([10,3,0,1]);
-2
```

A.3.4: SecondEigenvalueFromSRGParameters

▶ `SecondEigenvalueFromSRGParameters([v,k,a,b])` *(function)*

Returns: An integer or an element of a cyclotomic field.
Given feasible strongly regular graph parameters `[v, k, a, b]`, this function returns the second largest eigenvalue of a strongly regular graph with parameters (`v,k,a,b`). If the eigenvalue is integer, the object returned is an integer. If the eigenvalue is irrational, the object returned lies in a cyclotomic field.

e.g.

```
gap> SecondEigenvalueFromSRGParameters([5,2,0,1]);
E(5)+E(5)^4
gap> SecondEigenvalueFromSRGParameters([10,3,0,1]);
1
```

A.3.5: LeastEigenvalueMultiplicity

▸ LeastEigenvalueMultiplicity([v,k,a,b]) (function)

Returns: An integer.

Given feasible strongly regular graph parameters [v, k, a, b] this function returns the multiplicity of the least eigenvalue of a strongly regular graph with parameters (v,k,a,b).

e.g.

```
gap> LeastEigenvalueMultiplicity([16,9,4,6]);
6
gap> LeastEigenvalueMultiplicity([25,12,5,6]);
12
```

A.3.6: SecondEigenvalueMultiplicity

▸ SecondEigenvalueMultiplicity([v,k,a,b) (function)

Returns: An integer.

Given feasible strongly regular graph parameters [v, k, a, b] this function returns the multiplicity of the second eigenvalue of a strongly regular graph with parameters (v,k,a,b).

e.g.

```
gap> SecondEigenvalueMultiplicity([16,9,4,6]);
9
gap> SecondEigenvalueMultiplicity([25,12,5,6]);
12
```

A.4 Regular induced subgraphs

In this section we give methods to investigate regular induced subgraphs of regular graphs.

Spectral bounds

In this section, we introduce some bounds on regular induced subgraphs of regular graphs, which depend on the spectrum of the graph.

A.4.1: HoffmanCocliqueBound

- HoffmanCocliqueBound(*gamma*) (operation)
- HoffmanCocliqueBound(*parms*) (operation)

Returns: An integer.

Given a non-null regular graph ***gamma***, this function returns the Hoffman coclique bound of ***gamma***.

Given feasible strongly regular graph parameters ***parms***, this function returns the Hoffman coclique bound of a strongly regular graph with parameters ***parms***.

Let Γ be a non-null regular graph with parameters (v, k) and least eigenvalue s. The *Hoffman coclique bound*, or *ratio bound* of Γ, is defined as

$$\delta = \lfloor \left(\frac{v}{k - s} \right) \rfloor.$$

It is known that any coclique in Γ must contain at most δ vertices (see Brouwer and Haemers [15]).

e.g.

```
gap> HoffmanCocliqueBound(HammingGraph(3,5));
25
gap> HoffmanCocliqueBound(HammingGraph(2,5));
5
gap> parms:=SRGParameters(HammingGraph(2,5));
[ 25, 8, 3, 2 ]
gap> HoffmanCocliqueBound(parms);
5
```

A.4.2: HoffmanCliqueBound

▸ `HoffmanCliqueBound(gamma)` *(operation)*
▸ `HoffmanCliqueBound(parms)` *(operation)*

Returns: An integer.

Given a non-null, non-complete regular graph ***gamma***, this function returns the Hoffman clique bound of ***gamma***.

Given feasible strongly regular graph parameters ***parms***, this function returns the Hoffman clique bound of a strongly regular graph with parameters ***parms***.

Let Γ be a non-null, non-complete regular graph. The *Hoffman clique bound* of Γ, is defined as the Hoffman coclique bound of its complement (see `HoffmanCocliqueBound` (A.4.1)). It is known that the Hoffman clique bound is an upper bound on the number of vertices in any clique of Γ (see Brouwer and Haemers [15]). Note that in the case that Γ is a strongly regular graph, this function returns the value of the well-known *Delsarte-Hoffman clique bound* (see Delsarte [27]).

e.g.

```
gap> gamma:=EdgeOrbitsGraph(CyclicGroup(IsPermGroup,15),
[[1,2],[2,1]]);;
gap> HoffmanCliqueBound(gamma);
2
gap> gamma:=JohnsonGraph(7,2);;
gap> HoffmanCliqueBound(gamma);
6
gap> parms:=SRGParameters(gamma);
[ 21, 10, 5, 4 ]
gap> HoffmanCliqueBound(parms);
6
```

A.4.3: HaemersRegularUpperBound

▸ `HaemersRegularUpperBound(gamma, d)` *(operation)*
▸ `HaemersRegularUpperBound(parms, d)` *(operation)*

Returns: An integer.

Given a non-null regular graph ***gamma*** and non-negative integer ***d***, this function returns the Haemers upper bound on ***d***-regular induced subgraphs of ***gamma***.

Given feasible strongly regular graph parameters ***parms*** and non-negative integer ***d***, this function returns the Haemers upper bound on ***d***-regular induced subgraphs of a strongly regular graph with parameters ***parms***.

Let Γ be a non-null regular graph with parameters (v,k) and least eigenvalue s and let d be a non-negative integer. The *Haemers upper bound* on d-regular induced subgraphs of Γ, is defined as

$$\delta = \lfloor\left(\frac{v(d-s)}{k-s}\right)\rfloor.$$

It is known that any d-regular induced subgraph in Γ has order at most δ (see Chapter 3).

e.g.

```
gap> HaemersRegularUpperBound(SimsGerwitzGraph(),3);
28
gap> HaemersRegularUpperBound([56,10,0,2],0);
16
```

A.4.4: HaemersRegularLowerBound

```
▸ HaemersRegularLowerBound(gamma, d)                    (operation)
▸ HaemersRegularLowerBound(parms, d)                    (operation)
```

Returns: An integer.

Given a connected regular graph ***gamma*** and non-negative integer ***d***, this function returns the Haemers lower bound on ***d***-regular induced subgraphs of ***gamma***.

Given the parameters of a connected strongly regular graph, ***parms***, and non-negative integer ***d***, this function returns the Haemers lower bound on ***d***-regular induced subgraphs of a strongly regular graph with parameters *parms*.

Let Γ be a connected regular graph with parameters (v, k) and second eigenvalue r and let d be a non-negative integer. The *Haemers lower bound* on d-regular induced subgraphs of Γ, is defined as

$$\delta = \lfloor\left(\frac{v(d-r)}{k-r}\right)\rfloor.$$

It is known that any d-regular induced subgraph in Γ has order at least δ (see Chapter 3).

e.g.

```
gap> HaemersRegularLowerBound(HoffmanSingletonGraph(),4);
20
gap> HaemersRegularLowerBound([50,7,0,1],3);
10
```

Block intersection polynomials and bounds

In this section, we introduce functions related to the block intersection polynomials, defined in Soicher [53]. If you would like to know more about the properties of these polynomials, see Soicher [53] and [54], and Greaves and Soicher [38].

A.4.5: CliqueAdjacencyPolynomial

```
▸ CliqueAdjacencyPolynomial(parms, x, y)                    (function)
```

Returns: A polynomial.

Given feasible edge-regular graph parameters *parms* and indeterminates *x*,*y*, this function returns the clique adjacency polynomial with respect to the parameters *parms* and indeterminates *x*,*y*, defined in Soicher [54].

Let Γ be an edge-regular graph with parameters (v,k,a). The *clique adjacency polynomial* of Γ is defined as

$$C(x,y) := x(x+1)(v-y) - 2xy(k-y+1) + y(y-1)(a-y+2).$$

e.g.

```
gap> x:=Indeterminate(Rationals,"x");
x
gap> y:=Indeterminate(Rationals,"y");
y
gap> CliqueAdjacencyPolynomial([21,8,3],x,y);
-x^2*y-y^3+21*x^2-x*y+8*y^2+21*x-23*y
```

A.4.6: CliqueAdjacencyBound

▸ `CliqueAdjacencyBound(parms)` *(function)*

Returns: An integer.

Given feasible edge-regular graph parameters ***parms***, this function returns the clique adjacency bound with respect to the parameters ***parms***, defined in Soicher [53].

Let Γ be an edge-regular graph with parameters (v,k,a), and let C be its corresponding clique adjacency poylnomial (see `CliqueAdjacencyPolynomial` (A.4.5)). The *clique adjacency bound* of Γ is defined as the smallest integer $y \geq 2$ such that there exists an integer m for which $C(m,y+1) < 0$. It is known that the clique adjacency bound is an upper bound on the number of vertices in any clique of Γ.

e.g.

```
gap> CliqueAdjacencyBound([16,6,2]);
4
```

A.4.7: RegularAdjacencyPolynomial

▸ `RegularAdjacencyPolynomial(parms, x, y, d)` *(function)*

Returns: A polynomial.

Given feasible strongly regular graph parameters ***parms*** and indeterminates ***x,y,d***, this function returns the regular adjacency polynomial with respect to the parameters ***parms*** and indeterminates ***x,y,d***, as defined in Chapter 3.

Let Γ be a strongly regular graph with parameters (v, k, a, b). The *regular adjacency polynomial* of Γ is defined as

$$R(x, y, d) := x(x+1)(v-y) - 2xyk + (2x+a-b+1)yd + y(y-1)b - yd^2.$$

e.g.

```
gap> RegularAdjacencyPolynomial([16,6,2,2],"x","y","d");
-x^2*y+2*x*y*d-y*d^2+16*x^2-x*y+2*y^2+y*d+4*x-2*y
```

A.4.8: RegularAdjacencyUpperBound

▸ `RegularAdjacencyUpperBound(parms,d)` *(function)*

Returns: An integer.

Given strongly regular graph parameters ***parms*** and non-negative integer ***d***, this function returns the regular adjacency upper bound with respect to the parameters ***parms*** and integer ***d***, defined in Chapter 3.

Let Γ be a strongly regular graph with parameters (v, k, a, b), and let R be its corresponding regular adjacency poylnomial (see `RegularAdjacencyPolynomial` (A.4.7)). For fixed d, the *regular adjacency upper bound* of Γ is defined as the largest integer $d + 1 \leq y \leq v$ such that for all integers m, we have $R(m, y, d) \geq 0$ if such a y exists, and 0 otherwise. It is known that the regular adjacency upper bound is an upper bound on the number of vertices in any d-regular induced subgraph of Γ.

e.g.

```
gap> RegularAdjacencyUpperBound([56,10,0,2],3);
28
```

A.4.9: RegularAdjacencyLowerBound

▸ `RegularAdjacencyLowerBound(parms,d)` *(function)*

Returns: An integer.

Given the parameters of a connected strongly regular graph, ***parms***, and non-negative integer ***d***, this function returns the regular adjacency lower bound with respect to the parameters ***parms*** and integer ***d***, defined in Chapter 3.

Let Γ be a strongly regular graph with parameters (v, k, a, b), and let R be its corresponding regular adjacency poylnomial (see `RegularAdjacencyPolynomial` (A.4.7)). For fixed d, the *regular adjacency lower bound* of Γ is defined as the smallest integer $d + 1 \leq y \leq v$ such that for all integers m, we have $R(m, y, d) \geq 0$ if such a y, and $v + 1$ otherwise. It is known that the regular adjacency lower bound is a lower bound on the number of vertices in any d-regular induced subgraph of Γ.

e.g.

```
gap> RegularAdjacencyLowerBound([50,7,0,1],2);
5
```

Regular sets

In this section we give functions to investigate regular sets, with a focus on regular sets in strongly regular graphs.

A.4.10: Nexus

▸ `Nexus(gamma, U)` *(function)*

Returns: An integer or `fail`.

Given a graph ***gamma*** and a subset ***U*** of its vertices, this function returns the nexus of ***U***. If ***U*** is not an m-regular set for some $m > 0$, the function returns `fail`.

e.g.

```
gap> Nexus(SquareLatticeGraph(5),[1,2,3,4,6]);
fail
gap> Nexus(SquareLatticeGraph(5),[1,2,3,4,5]);
1
```

A.4.11: RegularSetParameters

▸ `RegularSetParameters(gamma, U)` (function)

Returns: A list or `fail`.

Given a graph ***gamma*** and a subset ***U*** of its vertices, this function returns a list ***[d,m]*** such that ***U*** is a (***d,m***)-regular set. If ***U*** is not an (d, m)-regular set for some d, m, the function returns `fail`.

e.g.

```
gap> RegularSetParameters(SquareLatticeGraph(5),[6,11,16,21]);
fail
gap> RegularSetParameters(SquareLatticeGraph(5),[1,6,11,16,21]);
[ 4, 1 ]
```

A.4.12: IsRegularSet

▸ `IsRegularSet(gamma, U, opt)` (function)

Returns: `true` or `false`.

Given a graph ***gamma*** and a subset ***U*** of its vertices, this function returns `true` if ***U*** is a regular set, and `false` otherwise.

The input ***opt*** should take a boolean value `true` or `false`. This option effects the output of the function in the following way.

true this function will return **true** if and only if *U* is a (d, m)-regular set for some d, m.

false this function will return **true** if and only if *U* is a m-regular set for some m.

e.g.

```
gap> IsRegularSet(HoffmanSingletonGraph(),[11..50],false);
true
gap> IsRegularSet(HoffmanSingletonGraph(),[11..50],true);
false
```

A.4.13: RegularSetSRGParameters

▶ `RegularSetSRGParameters(parms, d)` (function)

Returns: A list.

Given feasible strongly regular graph parameters *parms* and non-negative integer *d*, this function returns a list of pairs *[s,m]* with the following properties:

- (*d,m*) are feasible regular set parameters for a strongly regular graph with parameters *parms*.
- *s* is the order of any (*d,m*)-regular set in a strongly regular graph with parameters *parms*.

Let Γ be a strongly regular graph with parameters (v, k, a, b) and let R be its corresponding regular adjacency polynomial (see `RegularAdjacencyPolynomial` (A.4.7)). Then the tuple (d, m) is a *feasible regular set parameter tuple* for Γ if d, m are non-negative integers and there exists a positive integer s such that

$$R(m-1, s, d) = R(m, s, d) = 0.$$

It is known that any (d, m)-regular set of size s in Γ must satisfy the above conditions (see Chapter 3).

e.g.

```
gap> RegularSetSRGParameters([16,6,2,2],4);
[ [ 8, 2 ], [ 12, 6 ] ]
```

Neumaier graphs

In this section, we give functions to investigate regular cliques in edge-regular graphs.

A.4.14: NGParameters

▶ `NGParameters(`*gamma*`)` *(function)*

Returns: A list or `fail`.

Given a graph *gamma*, this function returns the Neumaier graph parameters of *gamma*. If *gamma* is not a Neumaier graph, the function returns `fail`.

e.g.

```
gap> NGParameters(HigmanSimsGraph());
fail
gap> NGParameters(TriangularGraph(10));
[ [ 45, 16, 8, 9, 2 ] ]
```

A.4.15: IsNG

▶ `IsNG(`*gamma*`)` *(function)*

Returns: `true` or `false`.

Given a graph *gamma*, this function returns `true` if *gamma* is a Neumaier graph, and `false` otherwise.

e.g.

```
gap> IsNG(HammingGraph(3,7));
false
gap> IsNG(HammingGraph(2,7));
true
```

A.4.16: IsFeasibleNGParameters

▸ IsFeasibleNGParameters(*[v,k,a,s,m]*) *(function)*

Returns: `true` or `false`.

Given a list of integers of length 5, *[v,k,a,s,m]*, this function returns `true` if (*v,k,a;s,m*) is a feasible parameter tuple for a Neumaier graph. Otherwise, the function returns `false`.

The tuple $(v, k, a; s, m)$ is a *feasible* parameter tuple for a Neumaier graph if it satisfies the following conditions:

- (v, k, a) is a feasible edge-regular graph parameter tuple;
- $0 < m \leq s$ and $2 \leq s \leq a + 2$;
- $(v - s)m = (k - s + 1)s$;
- $(k - s + 1)(m - 1) = (a - s + 2)(s - 1)$.

Any Neumaier graph must have parameters which satisfy these conditions (see Chapter 4).

e.g.

```
gap> IsFeasibleNGParameters([35,16,6,5,2]);
true
gap> IsFeasibleNGParameters([37,18,8,5,2]);
false
```

A.4.17: RegularCliqueERGParameters

▸ `RegularCliqueERGParameters(parms)` *(function)*

Returns: A list.

Given feasible edge-regular graph parameters `parms=[v,k,a]`, this function returns a list of pairs `[s,m]`, such that (`v,k,a;s,m`) are feasible Neumaier graph parameters (as defined in `IsFeasibleNGParameters` (A.4.16)).

e.g.

```
gap> RegularCliqueERGParameters([8,7,6]);
[ [ 1, 1 ], [ 2, 2 ], [ 3, 3 ], [ 4, 4 ], [ 5, 5 ], [ 6, 6 ],
  [ 7, 7 ] ]
gap> RegularCliqueERGParameters([8,6,4]);
[ [ 4, 3 ] ]
gap> RegularCliqueERGParameters([16,9,4]);
[ [ 4, 2 ] ]
```

A.5 Strongly regular graphs

In this section we give functions to investigate strongly regular graphs. In particular, we provide a collection of strongly regular graphs which can be a useful computational resource.

Strongly regular graph parameter tuples

In this section, we give functions to investigate the parameters of a strongly regular graph. For the definition of feasible strongly regular graph parameters, see `IsFeasibleSRGParameters` (A.2.9).

A.5.1: ComplementSRGParameters

▸ `ComplementSRGParameters(parms)` *(function)*

Returns: A list.

Given feasible strongly regular graph parameters ***parms***, this function returns the complement parameters of ***parms***.

Suppose Γ is a strongly regular graph with parameters (v, k, a, b). Then the complement of Γ is a strongly regular graph with parameters $(v, v-k-1, v-2-2k+b, v-2k+a)$ (see Brouwer, Cohen and Neumaier [14]). We define the *complement parameters* of the feasible strongly regular graph parameter tuple (v, k, a, b) as the tuple $(v, v-k-1, v-2-2k+b, v-2k+a)$.

e.g.

```
gap> ComplementSRGParameters([16,9,4,6]);
[ 16, 6, 2, 2 ]
```

A.5.2: SRGToGlobalParameters

▸ `SRGToGlobalParameters(parms)` *(function)*

Returns: A list.

Given feasible strongly regular graph parameters ***parms***, this function returns the global parameters of a graph with strongly regular graph parameters ***parms***. For information on global parameters of a graph, see the GRAPE manual [55].

e.g.

```
gap> SRGToGlobalParameters([50,7,0,1]);
[ [ 0, 0, 7 ], [ 1, 0, 6 ], [ 1, 6, 0 ] ]
```

A.5.3: GlobalToSRGParameters

▸ `GlobalToSRGParameters(parms)` *(function)*

Returns: A list.

Given the global parameters ***parms*** of a connected strongly regular graph, this function returns the strongly regular graph parameters of the graph. For information on global parameters of a graph, see the GRAPE manual [55].

e.g.

```
gap> parms:=GlobalParameters(JohnsonGraph(5,3));
[ [ 0, 0, 6 ], [ 1, 3, 2 ], [ 4, 2, 0 ] ]
gap> GlobalToSRGParameters(parms);
[ 10, 6, 3, 4 ]
```

A.5.4: IsPrimitiveSRGParameters

▸ `IsPrimitiveSRGParameters([v,k,a,b])` *(function)*

Returns: `true` or `false`.

Given a list of integers of length 4, `[v,k,a,b]`, this function returns `true` if (`v,k,a,b`) is a feasible parameter tuple for a primitive strongly regular graph. Otherwise, this function returns `false`.

Let (v, k, a, b) be feasible strongly regular parameters with complement parameters (v', k', a', b'). Then the parameter tuple (v, k, a, b) is called *primitive* if $b \neq 0$ and $b' \neq 0$.

A strongly regular graph Γ is called *primitive* if Γ and its complement is connected. It is known that a non-primitive strongly regular graph is a union of cliques, or the complement of a union of cliques. From our definition, it follows that a strongly regular graph Γ is primitive if and only if Γ has primitive strongly regular graph parameters (see Brouwer, Cohen and Neumaier [14]).

e.g.

```
gap> IsFeasibleSRGParameters([8,6,4,6]);
true
gap> IsPrimitiveSRGParameters([8,6,4,6]);
false
gap> IsPrimitiveSRGParameters([10,6,3,4]);
true
```

A.5.5: IsTypeISRGParameters

▸ `IsTypeISRGParameters([v,k,a,b])` *(function)*

Returns: `true` or `false`.

Given a list of integers of length 4, `[v,k,a,b]`, this function returns `true` if (`v,k,a,b`) is a feasible parameter tuple for a type I strongly regular graph. Otherwise, this function returns `false`.

A feasible strongly regular parameter tuple (v, k, a, b) is of *type I* (or a *conference graph*) if there exists a positive integer t such that $v = 4t + 1, k = 2t, a = t - 1, b = t$.

There are two types of strongly regular graphs, called type I and type II (see Brouwer, Cohen and Neumaier [14]). Let Γ be a strongly regular graph with parameters (v, k, a, b). Then Γ is of *type I* if and only if (v, k, a, b) is of type I.

e.g.

```
gap> IsTypeISRGParameters([5,2,0,1]);
true
gap> IsTypeISRGParameters([9,4,1,2]);
true
gap> IsTypeISRGParameters([10,3,0,1]);
false
```

A.5.6: IsTypeIISRGParameters

▸ `IsTypeIISRGParameters([v,k,a,b])` *(function)*

Returns: `true` or `false`.

Given a list of integers of length 4, `[v,k,a,b]`, this function returns `true` if (`v,k,a,b`) is a feasible parameter tuple for a type II strongly regular graph. Otherwise, this function returns `false`.

A feasible strongly regular parameter tuple (v, k, a, b) is of *type II* if the polynomial $x^2 - (a - b)x + b - k$ has integer zeros.

There are two types of strongly regular graphs, called type I and *type II* (see Brouwer, Cohen and Neumaier [14]). Let Γ be a strongly regular graph with parameters (v,k,a,b). Then Γ is of *type II* if and only if all of its eigenvalues are integer. The eigenvalues of Γ are k and the zeros of the polynomial $x^2-(a-b)x+b-k$ (see Brouwer, Cohen and Neumaier [14]). From our definition, it follows that Γ is of type II if and only if (v,k,a,b) is of type II.

e.g.

```
gap> IsTypeIISRGParameters([5,2,0,1]);
false
gap> IsTypeIISRGParameters([9,4,1,2]);
true
gap> IsTypeIISRGParameters([10,3,0,1]);
true
```

A.5.7: KreinParameters

▶ `KreinParameters([v,k,a,b])` *(function)*

Returns: A list.

Given feasible strongly regular graph parameters `[v, k, a, b]`, this function returns a list of (non-trivial) Krein parameters of a strongly regular graph with parameters (`v,k,a,b`).

If the eigenvalues of a strongly regular graph are integer, the object returned is a list of integers. If the eigenvalues are irrational, the object returned will be a list of cyclotomic numbers.

Let Γ be a strongly regular graph with parameters (v,k,a,b) and eigenvalues $k\geq r>s$. Then the *Krein parameters* of Γ are the numbers

$$K_1=(k+r)(s+1)^2-(r+1)(k+r+2rs),$$

$$K_2=(k+s)(r+1)^2-(s+1)(k+s+2rs).$$

For information on the Krein parameters of strongly regular graphs, see Brouwer, Cohen and Neumaier [14].

e.g.

```
gap> KreinParameters([10,6,3,4]);
[ 1, 16 ]
gap> KreinParameters([13,6,2,3]);
[ -14*E(13)-12*E(13)^2-14*E(13)^3-14*E(13)^4-12*E(13)^5-12*E(13)^6
-12*E(13)^7-12*E(13)^8-14*E(13)^9-14*E(13)^10-12*E(13)^11
-14*E(13)^12,
-12*E(13)-14*E(13)^2-12*E(13)^3-12*E(13)^4-14*E(13)^5-14*E(13)^6
-14*E(13)^7-14*E(13)^8-12*E(13)^9-12*E(13)^10-14*E(13)^11
-12*E(13)^12 ]
```

A.5.8: IsKreinConditionsSatisfied

▸ `IsKreinConditionsSatisfied([v,k,a,b])` (function)

Returns: `true` or `false`.

Given feasible strongly regular graph parameters `[v, k, a, b]`, this function returns `true` if the parameters satisfy the Krein conditions. Otherwise, this function returns `false`.

Let Γ be a strongly regular graph with parameters (v,k,a,b) and Krein parameters K_1, K_2 (see `KreinParameters` (A.5.7). The *Krein conditions* of Γ are the inequalities

$$K_1 \geq 0, K_2 \geq 0.$$

It is known that any strongly regular graph must have parameters that satisfy the Krein conditions. For information on the Krein conditions of strongly regular graphs, see Brouwer, Cohen and Neumaier [14].

e.g.

```
gap> IsKreinConditionsSatisfied([28,9,0,4]);
false
gap> IsKreinConditionsSatisfied([13,6,2,3]);
true
gap> IsKreinConditionsSatisfied([10,6,3,4]);
true
```

A.5.9: IsAbsoluteBoundSatisfied

▶ `IsAbsoluteBoundSatisfied([v,k,a,b])` *(function)*

Returns: `true` or `false`.

Given primitive strongly regular graph parameters `[v, k, a, b]`, this function returns `true` if the parameters satisfy the absolute bound. Otherwise, this function returns `false`.

Let Γ be a primitive strongly regular graph with parameters (v, k, a, b) and eigenvalues $k \geq r > s$, with multiplicities $1, f, g$. The *absolute bound* for the number of vertices of Γ is

$$v \leq f(f+3)/2, v \leq g(g+3)/2.$$

For information on the absolute bound of strongly regular graphs, see Brouwer, Cohen and Neumaier [14].

e.g.

```
gap> IsAbsoluteBoundSatisfied([13,6,3,4]);
false
gap> IsAbsoluteBoundSatisfied([50,21,4,12]);
false
gap> IsAbsoluteBoundSatisfied([50,21,8,9]);
true
```

Small strongly regular graphs

In this section, we give functions to access and use the library of strongly regular graphs currently stored in this package. The information on small strongly regular graphs in this section is based on the tables of Andries Brouwer [12]. The strongly regular graphs were either constructed directly by the author, or collected from the website of Ted Spence [57].

A.5.10: AGT_Brouwer_Parameters_MAX

▸ `AGT_Brouwer_Parameters_MAX` (variable)

This variable stores the largest value v for which the current package contains information about the parameters of all strongly regular graphs with at most v vertices. This information is stored in the list `AGT_Brouwer_Parameters` (A.5.11).

A.5.11: AGT_Brouwer_Parameters

▸ `AGT_Brouwer_Parameters` (variable)

This variable stores information about the feasible strongly regular graph parameters found in Brouwer [12] and the available strongly regular graphs. *`AGT_Brouwer_Parameters`* is a list, each element of which is a list of length 4. For an element `[parms,status,stored,num]`, each entry denotes the following;

`parms` A feasible strongly regular graph parameter tuple `[v,k,a,b]`, where `v` is less than `AGT_Brouwer_Parameters_MAX` (A.5.10).

`status` the status of the known strongly regular graphs with parameters `parms`. This entry is

- 0 if there exists a strongly regular graph with parameters `parms`, and all such graphs have been enumerated,
- 1 if there exists a strongly regular graph with parameters `parms`, but all such graphs have not been enumerated,

- 2 if it is not known if a strongly regular graph with parameters parms exists,
- 3 if it has been proven that no strongly regular graph with parameters parms exists.

stored the number of non-isomorphic strongly regular graphs with parameters parms that are available in the current package.

num the number of non-isomorphic strongly regular graphs with parameters parms that exist. If this has not been determined, the value of num is set to -1.

e.g.

```
gap> AGT_Brouwer_Parameters[34];
[ [ 36, 20, 10, 12 ], 0, 32548, 32548 ]
gap> AGT_Brouwer_Parameters[35];
[ [ 37, 18, 8, 9 ], 1, 6760, -1 ]
gap> AGT_Brouwer_Parameters[2530];
[ [ 765, 176, 28, 44 ], 2, 0, -1 ]
gap> AGT_Brouwer_Parameters[4530];
[ [ 1293, 646, 322, 323 ], 3, 0, 0 ]
```

A.5.12: IsSRGAvailable

▶ `IsSRGAvailable(`*parms*`)` (function)

Returns: true or false.

Given feasible strongly regular graph parameters *parms*, this function returns true if there is a strongly regular graph with parameters *parms* stored in this package. If *parms* is a feasible parameter tuple but there is no strongly regular graphs with parameters *parms* available, the function returns false.

e.g.

```
gap> IsSRGAvailable([28,12,6,4]);
true
gap> IsSRGAvailable([28,9,0,4]);
false
```

A.5.13: SRGLibraryInfo

▶ SRGLibraryInfo(*parms*) (function)

Returns: A list.

Given feasible strongly regular graph parameters ***parms***, with first parameter ***v*** at most `AGT_Brouwer_Parameters_MAX` (A.5.10), this function returns the element of `AGT_Brouwer_Parameters` (A.5.11) corresponding to ***parms***.

e.g.

```
gap> SRGLibraryInfo([37,18,8,9]);
[ [ 37, 18, 8, 9 ], 1, 6760, -1 ]
gap> SRGLibraryInfo([36,15,6,6]);
[ [ 36, 15, 6, 6 ], 0, 32548, 32548 ]
```

A.5.14: SRG

▶ SRG(*parms*) (function)

Returns: A record or `fail`.

Given feasible strongly regular graph parameters ***parms*** and positive integer ***n***, this function returns the ***n***th strongly regular graph with parameters ***parms*** available in this package. If there there is no ***n***th strongly regular graph with parameters ***parms*** available, the function returns `fail`.

e.g.

```
gap> SRG([16,6,2,2],1);
rec( adjacencies := [ [ 2, 3, 4, 5, 6, 7 ] ],
group := <permutation group with 6 generators>, isGraph := true,
names := [ 1 .. 16 ], order := 16, representatives := [ 1 ],
schreierVector := [ -1, 6, 4, 3, 5, 5, 5, 6, 6, 6, 4, 4, 4, 3, 3,
 3 ] )
gap> SRG([16,6,2,2],2);
rec( adjacencies := [ [ 2, 3, 4, 5, 6, 7 ] ],
group := Group([ (3,4)(5,6)(8,9)(11,14)(12,13)(15,16),
 (2,3)(4,5)(6,7)(9,11)(10,12)(14,15), (1,2)(5,8)(6,9)
(7,10)(11,12)(13,14) ]), isGraph := true, names := [ 1 .. 16 ],
order := 16, representatives := [ 1 ],
schreierVector := [ -1, 3, 2, 1, 2, 1, 2, 3, 3, 3, 2, 2, 1, 1, 2,
 1 ] )
gap> SRG([28,9,0,4],1);
fail
```

A.5.15: NrSRGs

▶ NrSRGs(*parms*) (function)

Returns: An integer.

Given feasible strongly regular graph parameters ***parms***, this function returns the number of pairwise non-isomorphic strongly regular graphs with parameters ***parms*** currently stored in this package.

e.g.

```
gap> NrSRGs([36,15,6,6]);
32548
gap> NrSRGs([28,9,0,4]);
0
```

A.5.16: OneSRG

▶ OneSRG(*parms*) (function)

Returns: A record or `fail`.

Given feasible strongly regular graph parameters ***parms***, this function returns one strongly regular graph with parameters ***parms***. If there there is no strongly regular graph with parameters ***parms*** available, the function returns `fail`.

e.g.

```
gap> OneSRG([16,9,4,6]);
rec( adjacencies := [ [ 8, 9, 10, 11, 12, 13, 14, 15, 16 ] ],
group := Group([ (6,7)(9,10)(12,13)(15,16),
(5,6)(8,9)(11,12)(14,15), (2,5)(3,6)(4,7)(9,11)(10,14)(13,15),
(1,2)(5,8)(6,9)(7,10) ]),
isGraph := true, names := [ 1 .. 16 ], order := 16,
representatives := [ 1 ],
schreierVector := [ -1, 4, 3, 3, 3, 2, 1, 4, 4, 4, 3, 2, 1, 3, 2,
1 ] )
gap> OneSRG([28,9,0,4]);
false
gap> OneSRG([21,9,0,4]);
fail
```

A.5.17: AllSRGs

▶ `AllSRGs(`*`parms`*`)` *(function)*

Returns: A list.

Given feasible strongly regular graph parameters ***parms***, this function returns a list of all pairwise non-isomorphic strongly regular graphs with parameters ***parms*** available in this package.

e.g.

```
gap> AllSRGs([16,6,2,2]);
[ rec( adjacencies := [ [ 2, 3, 4, 5, 6, 7 ] ],
group := <permutation group with 6 generators>, isGraph := true,
names := [ 1 .. 16 ], order := 16, representatives := [ 1 ],
schreierVector := [ -1, 6, 4, 3, 5, 5, 5, 6, 6, 6, 4, 4, 4, 3, 3,
3 ] ),
rec( adjacencies := [ [ 2, 3, 4, 5, 6, 7 ] ],
group := Group([ (3,4)(5,6)(8,9)(11,14)(12,13)(15,16),
 (2,3)(4,5)(6,7)(9,11)(10,12)(14,15), (1,2)
(5,8)(6,9)(7,10)(11,12)(13,14) ]), isGraph := true,
names := [ 1 .. 16 ], order := 16, representatives := [ 1 ],
schreierVector := [ -1, 3, 2, 1, 2, 1, 2, 3, 3, 3, 2, 2, 1, 1, 2,
1 ] )]
```

A.5.18: SRGIterator

▸ `SRGIterator(parms)` *(function)*

Returns: An iterator.

Given feasible strongly regular graph parameters ***parms***, this function returns an iterator of all pairwise non-isomorphic strongly regular graph with parameters ***parms*** that are stored in this package.

e.g.

```
gap> SRGIterator([16,6,2,2]);
<iterator>
```

A.5.19: SmallFeasibleSRGParameterTuples

▸ `SmallFeasibleSRGParameterTuples(v)` *(function)*

Returns: A list.

Given an integer *v*, this function returns a list of all feasible parameter tuples with at most *v* vertices, according to the list of Brouwer [12]. The list contains parameter tuples with first parameter at most `AGT_Brouwer_Parameters_MAX`

(A.5.10).

e.g.

```
gap> SmallFeasibleSRGParameterTuples(16);
[ [ 5, 2, 0, 1 ], [ 9, 4, 1, 2 ], [ 10, 3, 0, 1 ], [ 10, 6, 3, 4 ],
[ 13, 6, 2, 3 ], [ 15, 6, 1, 3 ], [ 15, 8, 4, 4 ], [ 16, 5, 0, 2 ],
[ 16, 10, 6, 6 ], [ 16, 6, 2, 2 ], [ 16, 9, 4, 6 ] ]
```

A.5.20: IsEnumeratedSRGParameterTuple

▶ IsEnumeratedSRGParameterTuple(*parms*) (function)

Returns: `true` or `false`.

Given feasible strongly regular graph parameters ***parms*** with first parameter *v* at most `AGT_Brouwer_Parameters_MAX` (A.5.10), this function returns `true` if the strongly regular graphs with parameters ***parms*** have been enumerated, according to the list of Brouwer [12]. Otherwise, this function returns `false`.

e.g.

```
gap> IsEnumeratedSRGParameterTuple([37,18,8,9]);
false
gap> IsEnumeratedSRGParameterTuple([56,10,0,2]);
true
```

A.5.21: IsKnownSRGParameterTuple

▶ IsKnownSRGParameterTuple(*parms*) (function)

Returns: `true` or `false`.

Given feasible strongly regular graph parameters ***parms*** with first parameter *v* at most `AGT_Brouwer_Parameters_MAX` (A.5.10), this function returns `true` if it is known that there exists a strongly regular graph with parameters ***parms***, according to the list of Brouwer [12]. Otherwise, this function returns `false`.

e.g.

```
gap> IsKnownSRGParameterTuple([64,28,12,12]);
true
gap> IsKnownSRGParameterTuple([64,30,18,10]);
false
gap> IsKnownSRGParameterTuple([65,32,15,16]);
false
```

A.5.22: IsAllSRGsStored

▸ `IsAllSRGsStored(parms)` *(function)*

Returns: `true` or `false`.

Given feasible strongly regular graph parameters ***parms*** with first parameter *v* at most `AGT_Brouwer_Parameters_MAX` (A.5.10), this function returns `true` if all pairwise non-isomorphic strongly regular graphs with parameters ***parms*** are stored in the package. Otherwise, this function returns `false`.

e.g.

```
gap> IsAllSRGsStored([37,18,8,9]);
false
gap> IsAllSRGsStored([36,15,6,6]);
true
```

Strongly regular graph constructors

In this section, we give functions to construct certain graphs, most of which are strongly regular graphs.

A.5.23: DisjointUnionOfCliques

▸ `DisjointUnionOfCliques([n1, n2, ...])` *(function)*

Returns: A record.

Given positive integers *n1, n2,...*, this function returns the graph consisting of the disjoint union of cliques with orders *n1, n2,...*.

e.g.

```
gap> DisjointUnionOfCliques(3,5,7);
rec( adjacencies := [ [ 2, 3 ], [ 5, 6, 7, 8 ],
[ 10, 11, 12, 13, 14, 15 ] ],
group := Group([ (1,2,3), (1,2), (4,5,6,7,8), (4,5),
(9,10,11,12,13,14,15), (9,10) ]), isGraph := true,
isSimple := true, order := 15, representatives := [ 1, 4, 9 ],
schreierVector := [ -1, 1, 1, -2, 3, 3, 3, 3, -3, 5, 5, 5, 5, 5, 5
] )
```

A.5.24: CompleteMultipartiteGraph

▸ CompleteMultipartiteGraph(*[n1, n2, ...]*) (function)

Returns: A record.

Given positive integers *n1, n2,...*, this function returns the complete multipartite graph with parts of orders *n1, n2,...*.

Let $n_1, n_2, \ldots, n_t$ be positive integers. Then the *complete multipartite graph*, $K_{n_1,n_2,\ldots,n_t}$, has vertex set that can be partitioned into t disjoint sets $X_1, X_2, \ldots, X_t$ of sizes $n_1, n_2, \ldots, n_t$ such that distinct vertices are adjacent if and only if they belong to different X_i.

e.g.

```
gap> CompleteMultipartiteGraph(4,2,1);
rec( adjacencies := [ [ 5, 6, 7 ], [ 1, 2, 3, 4, 7 ],
[ 1, 2, 3, 4, 5, 6 ] ], group := Group([ (1,2,3,4), (1,2), (5,6)
]),
isGraph := true, isSimple := true, order := 7,
representatives := [ 1, 5, 7 ],
schreierVector := [ -1, 1, 1, 1, -2, 3, -3 ] )
```

A.5.25: TriangularGraph

▶ `TriangularGraph(n)` (function)

Returns: A record.

Given an integer `n`, where `n` ≥ 3, this function returns the triangular graph on `n` points.

Let n be an integer, where $n \geq 3$. The *triangular graph* , $T(n)$, has vertex set consisting of the subsets of $\{1, 2, ..., n\}$ of size 2, and two distinct vertices A, B are joined by an edge precisely when $|A \cap B| = 1$.

The graph $T(n)$ is strongly regular with parameters $(\binom{n}{2}, 2(n-2), n-2, 4)$. For $n \neq 8$, $T(n)$ is the unique strongly regular graph with its parameters. There are four pairwise non-isomomorphic strongly regular graphs that have the same parameters as $T(8)$, which are the triangular graph $T(8)$ and the *Chang graphs* (see Connor [22] and Chang [20]).

e.g.

```
gap> TriangularGraph(7);
rec( adjacencies := [ [ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 ] ],
group := Group([
(1,7,12,16,19,21,6)(2,8,13,17,20,5,11)(3,9,14,18,4,10,15),
(2,7)(3,8)(4,9)(5,10)(6,11) ]), isGraph := true, isSimple := true,
names := [ [ 1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 1, 5 ], [ 1, 6 ],
[ 1, 7 ], [ 2, 3 ], [ 2, 4 ], [ 2, 5 ], [ 2, 6 ], [ 2, 7 ],
[ 3, 4 ], [ 3, 5 ], [ 3, 6 ], [ 3, 7 ], [ 4, 5 ], [ 4, 6 ],
[ 4, 7 ], [ 5, 6 ], [ 5, 7 ], [ 6, 7 ] ],
order := 21, representatives := [ 1 ],
schreierVector := [ -1, 2, 2, 2, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1,
1, 1, 1, 1, 1, 1 ] )
```

A.5.26: SquareLatticeGraph

▶ `SquareLatticeGraph(n)` (function)

Returns: A record.

Given an integer `n`, where `n` ≥ 2, this function returns the square lattice graph on n^2 points.

Let n be an integer, where $n \geq 2$. The *square lattice graph*, $L_2(n)$, has vertex set $\{1, 2, ..., n\} \times \{1, 2, ..., n\}$, and two distinct vertices are joined by an edge precisely when they have the same value at one coordinate.

The graph $L_2(n)$ is strongly regular with parameters $(n^2, 2(n-1), n-2, 2)$. For $n \neq 4$, $L_2(n)$ is the unique strongly regular graph with its parameters. There are two pairwise non-isomomorphic strongly regular graphs that have the same parameters as $L_2(4)$, which are the square lattice graph graph $L_2(4)$ and the *Shrikhande graph* (see Shrikhande [52]).

e.g.

```
gap> SquareLatticeGraph(6);
rec( adjacencies := [ [ 2, 3, 4, 5, 6, 7, 13, 19, 25, 31 ] ],
group := <permutation group with 5 generators>, isGraph := true,
names := [ [ 1, 1 ], [ 1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 1, 5 ],
[ 1, 6 ], [ 2, 1 ], [ 2, 2 ], [ 2, 3 ], [ 2, 4 ],
[ 2, 5 ], [ 2, 6 ], [ 3, 1 ], [ 3, 2 ], [ 3, 3 ],
[ 3, 4 ], [ 3, 5 ], [ 3, 6 ], [ 4, 1 ], [ 4, 2 ],
[ 4, 3 ], [ 4, 4 ], [ 4, 5 ], [ 4, 6 ], [ 5, 1 ],
[ 5, 2 ], [ 5, 3 ], [ 5, 4 ], [ 5, 5 ], [ 5, 6 ],
[ 6, 1 ], [ 6, 2 ], [ 6, 3 ], [ 6, 4 ],
[ 6, 5 ],
[ 6, 6 ] ],
order := 36, representatives := [ 1 ],
schreierVector :=
[ -1, 3, 3, 3, 3, 3, 1, 3, 3, 3, 3, 3, 1, 3, 3, 3, 3, 3,
1, 3, 3, 3, 3, 3, 1, 3, 3, 3, 3, 3, 1, 3, 3, 3, 3, 3 ] )
```

A.5.27: HoffmanSingletonGraph

▸ `HoffmanSingletonGraph()` (function)

Returns: A record.

This function returns the Hoffman-Singleton graph.

The *Hoffman-Singleton graph* is the unique strongly regular graph with parameters $(50, 7, 0, 1)$. For more information on this graph, see Brouwer [13].

e.g.

```
gap> gamma:=HoffmanSingletonGraph();;
```

A.5.28: HigmanSimsGraph

▶ HigmanSimsGraph() (function)

Returns: A record.

This function returns the Higman-Sims graph.

The *Higman-Sims graph* is the unique strongly regular graph with parameters $(100, 22, 0, 6)$. For more information on this graph, see Brouwer [13].

e.g.

```
gap> gamma:=HigmanSimsGraph();;
```

A.5.29: SimsGerwitzGraph

▶ SimsGerwitzGraph() (function)

Returns: A record.

This function returns the Sims-Gerwitz graph.

The *Sims-Gerwitz graph* is the unique strongly regular graph with parameters $(56, 10, 0, 2)$. For more information on this graph, see Brouwer [13].

e.g.

```
gap> gamma:=SimsGerwitzGraph();;
```

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